1. Introduction and Notation

Subatomic particles

elementary particles : no substructure
composite particles : composed of other particles

particles defined by mass, charge, spin, isospin, strangeness, chirality

2,2 u	1'280 c	173'000 t	0 g	125'000 H
d 4,7	s 96	b 4'180	γ 0	
0,511 e	106 μ	1'780 τ	91'200 Z	
ν_e ≈0	ν_μ ≈0	ν_τ ≈0	W 80'400	

Baryons :

$$\left.\begin{array}{l} p^+(uud),\ n^0(udd),\ \Lambda^0(uds),\ \Sigma^0(uds) \\ \Xi^0(uss),\ \Omega_c^0\ (ssc) \end{array}\right\} J^P = \frac{1}{2}^+$$

$$\left.\begin{array}{l} \Delta^0(uud),\ \Sigma^{*0}(uds),\ \Xi^{*0}(uss) \\ \Omega_c^{*0}\ (ssc) \end{array}\right\} J^P = \frac{3}{2}^+$$

Mesons :

$$\left.\pi^+(u\bar{d}),\ \eta\left(\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}\right),\ K^+(u\bar{s}),\ D^0(c\bar{u}),\ B^0(d\bar{b})\right\} J^P = 0^-$$

$$\left.\rho^0(u\bar{d}),\ \omega\left(\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}\right),\ J/\psi(c\bar{c}),\ K^{*0}(d\bar{s})\right\} J^P = 1^-$$

Symmetries

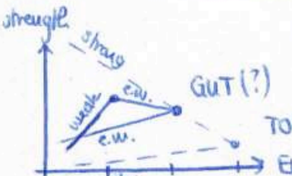
Continuous symmetries ⟷ conservation
- transl. in space - momentum
- transl. in time - energy
- rotation in space - ang. mom.

Discrete symmetries : C, P, T
Local/global gauge symmetries

Conserved quant.	Strong	E.M.	Weak	Symmetry
Energy + mom.	✓	✓	✓	transl. in spacetime
Electric charge	✓	✓	✓	U(1) local
Baryon no.	✓	✓	✓	U(1) global
lepton no.	✓	✓	✓	U(1) global
Isospin	✓	×	×	
Strangeness	✓	✓	×	
Parity	✓	✓	×	discrete
Charge conj.	✓	✓	×	discrete
Time reversal	✓	✓	×	discrete
CP	✓	✓	×	discrete
CPT	✓	✓	✓	discrete

Fundamental forces

Strong : 8 gluons, ~1, q
Electromagnetic : 1 photon, $\sim \frac{1}{137}$, ℓ/q
weak : $W^\pm/Z^0$, $\sim \frac{1}{40}$, ℓ/ν/q
Gravity : 1 graviton (?), weak, all

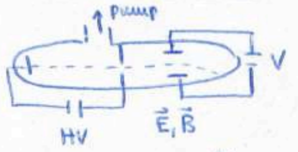


$\Lambda_{ew} \simeq 246$ GeV
$\Lambda_{GUT} \simeq 10^{16}$ GeV
$\Lambda_{Pl} \simeq \sqrt{\frac{\hbar c}{G}} = 1{,}2 \cdot 10^{19}$ GeV

Natural units

$c = \hbar = \varepsilon_0 = \mu_0 = 1$
$\hbar c = 197$ MeV·fm
$c = 2{,}998 \cdot 10^8$ m/s
$\hbar = 1{,}055 \cdot 10^{-34}$ Js
$e = 0{,}303$

Cathode ray tube



Thomson

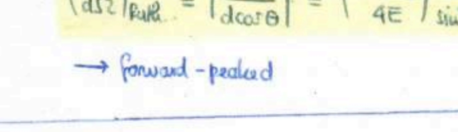
with $\vec{E}$ and $\vec{B}$: $e\frac{V}{d} = evB \rightarrow v = \frac{V}{Bd}$
Only with $\vec{E}$: $\delta(l) = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{e}{m}\right)\left(\frac{V}{d}\right)\left(\frac{l}{v}\right)^2$
Since $\left(\frac{e}{m}\right)_{\text{cathode ray}} << \left(\frac{e}{m}\right)_{\text{ions}}$ → electron

Rutherford's Scattering Experiment

Rutherford

→ consistent with the atom having a nucleus surrounded by electrons
→ nucleus of H-atom : proton

- ▶ interact. via Coulomb force (no magn. interact.)
- ▶ no recoil, classical eqs.



$$\frac{1}{r} = \frac{1}{b}\sin\alpha + \frac{D}{2b^2}(\cos\alpha - 1) \qquad D = \frac{\alpha z Z}{E} \qquad E = \frac{1}{2}mv^2$$

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{Ruth} = \left|\frac{b\,db}{d\cos\theta}\right| = \left(\frac{\alpha z Z}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

→ forward-peaked

2. Basic Concepts

Rates and Cross Sections

Flux φ : number of particles per unit time per unit surface

Cross section σ : $\left(\frac{d\sigma}{d\Omega}\right) = \frac{1}{\varphi}\frac{dN(\theta,\varphi)}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ $(d\sigma = b\,db\,d\varphi)$

Thin target approximation : $\varphi \longrightarrow$ 

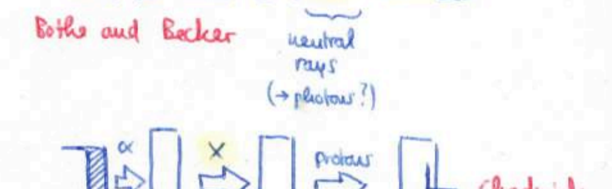
$R = N_{targets} \cdot \varphi \cdot \sigma$
$N(x) = N_0 e^{-\lambda x}$ with $\lambda = \frac{1}{\rho\sigma}$

Collinear and head-on collisions :  $R = \underbrace{n_1 n_2 |\vec{\beta}_1 - \vec{\beta}_2| V}_{=:\ F} \sigma$

Discovery of the neutron

$\alpha + Be \longrightarrow C + \underbrace{X}_{\text{neutral rays}}$ observed
(→ photons ?)

Bothe and Becker



Chadwick

$$\left.\begin{array}{l} M_p + E_\gamma = E_p' + E_\gamma' \\ \vec{p}_\gamma = \vec{p}_{p'} + \vec{p}_\gamma \end{array}\right\} E_\gamma = \frac{M_p(E_p' - M_p)}{M_p - E_p' + p_p'\cos\theta}$$

$$E_\gamma^{min} = \frac{M_p(E_p' - M_p)}{M_p - E_p' + p_p'} \approx 5\text{ MeV} \longrightarrow X = \text{neutrons}$$

4. Non-Relativistic Quantum Mechanics

First Quantization

Wavefunction $\Psi(\vec{x}_1, \dots, \vec{x}_N, t) \leftrightarrow |\Psi\rangle$ ← in Hilbert space; N fixed

$|\Psi(\vec{x},t)|^2 d^3\vec{x}$ is interpreted as a probability density

Observables: Hermitian Operators $\hat{A}^\dagger = \hat{A}$

$\hat{\vec{p}} = -i\vec{\nabla}$, $\hat{E} = +i\partial_t$, $[x_i, p_j] = i\delta_{ij}$

Schrödinger eq.: $i\partial_t\Psi(\vec{x},t) = \hat{H}\Psi(\vec{x},t) = \left(-\frac{1}{2m}\vec{\nabla}^2 + V(\vec{x},t)\right)\Psi(\vec{x},t)$

$$\rightarrow \partial_t \rho = \underbrace{-\frac{i}{2m}\vec{\nabla}\left[(\vec{\nabla}\Psi^*)\Psi - (\vec{\nabla}\Psi)\Psi^*\right]}_{-\vec{\nabla}\vec{j}}$$

Free particle: $N(\vec{x},t) = Ne^{-i(Et - \vec{p}\vec{x})} \rightarrow \rho = |N|^2, \vec{j} = \vec{v}|N|^2$

Schrödinger eq. with fields

$\hat{H} = \frac{1}{2m}(\vec{p} - e\vec{A}(\vec{x},t))^2 + e\Phi(\vec{x},t) + V(\vec{x},t)$

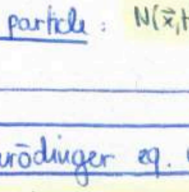
minimal substitution: $\vec{p} \rightarrow \vec{p} - e\vec{A}$, $E \rightarrow E - e\Phi$

$$\text{terms with } \vec{A}: \underbrace{\frac{ie}{m}\vec{A}\,\vec{\nabla}\Psi}_{\text{paramagnetic}} + \underbrace{\frac{ie}{2m}\Psi(\vec{\nabla}\vec{A})}_{\text{vanishes in Coulomb gauge}} + \underbrace{\frac{e^2}{2m}\vec{A}^2\Psi}_{\text{diamagnetic}}$$

→ uniform field $(\Phi = 0, \vec{\nabla}\vec{A} = 0)$: $\vec{A} = -\frac{1}{2}\vec{x} \times \vec{B}$ → terms with $\vec{A}$:

$$\underbrace{+\frac{e}{2m}(\vec{B}\cdot\vec{L})\Psi}_{\text{paramagn.}} + \underbrace{\frac{e^2}{8m}(\vec{x}\times\vec{B})^2\Psi}_{\text{diamagn.}}$$

Spin

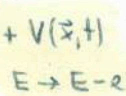


$\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$

→ expected: three spots for $m_z = -1, 0, 1$

→ measured: two spots for $m_s = -\frac{1}{2}, \frac{1}{2}$

Stern-Gerlach



Spin degree of freedom: $|\uparrow\rangle = |s, s_3\rangle = |\frac{1}{2}, \frac{1}{2}\rangle = \binom{1}{0}$

$|\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle = \binom{0}{1}$

$\vec{S} = \frac{1}{2}\vec{\sigma}$, $[S_i, S_j] = i\varepsilon_{ijk}S_k$, $S_\pm = S_x \pm iS_y$

$\hat{u}\cdot\vec{S}$ has eigenstates $|\chi_+\rangle = \begin{pmatrix}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\varphi}\end{pmatrix}$ $|\chi_-\rangle = \begin{pmatrix}-e^{-i\varphi}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\end{pmatrix}$

Pauli equation: $\hat{H} = \frac{1}{2m}\left(\vec{\sigma}(\vec{p} - e\vec{A})\right)^2 + e\phi\mathbb{1} = \frac{1}{2m}(\vec{p} - e\vec{A})^2\mathbb{1} + e\phi\mathbb{1} - \frac{e}{m}\vec{S}\vec{B}$

→ uniform field: $H = -\frac{1}{2m}\vec{\nabla}^2 - \frac{e}{2m}(\vec{L} + 2\vec{S})\vec{B} + \frac{e^2}{8m}(\vec{x}\times\vec{B})^2$

$\rightarrow \mu_s = -2\vec{S}\frac{e}{2m} \rightarrow g_e = -2$

Symmetries

Consider an infinitesimal transf. $U(\varepsilon) = \mathbb{1} + i\varepsilon G$ ← element of the Lie algebra

$\mathbb{1} \overset{!}{=} U(\varepsilon)^\dagger U(\varepsilon) \rightarrow G = G^\dagger$ Hermitian and the finite transformation is obtained via $U(\theta) = e^{i\theta G}$

$[H, U] = 0 \leftrightarrow [H, L] = 0$

Rotations:

$\Psi'(\vec{x}) = \Psi(R^{-1}\vec{x})$ with $R_z(\theta) = \begin{pmatrix}\cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}$

$|\Psi'\rangle = U(\theta)|\Psi\rangle$ in Hilbert space

$\Psi(R^{-1}(\delta\theta)\vec{x}) = \Psi(x - \delta\theta y, y + \delta\theta x, z) \simeq \Psi(x,y,z) + i\delta\theta L_z\Psi(x,y,z)$

$\rightarrow U(\varepsilon) = \mathbb{1} + i\delta\theta L_z \rightarrow U(\alpha) = e^{i\alpha(\hat{u}\cdot\vec{L})}$

Translations:

$\Psi'(x) = \Psi(\vec{x} - \vec{a})$

$|\Psi'\rangle = U(\vec{a})|\Psi\rangle$ in Hilbert space

$\Psi(\vec{x} - \delta\vec{a}) \simeq \Psi(\vec{x}) - i\delta\vec{\alpha}\hat{\vec{p}}\Psi(\vec{x}) \rightarrow U(\vec{a}) = e^{-i\vec{a}\vec{p}}$

Discrete Transformations:

P: $P = (-1)^l$ for Hydrogen; $P_\mu = -1$; $P(f) = +1$, $P(\bar{f}) = -1$ — $-\vec{x}, -\vec{p}, +\vec{L}, +\vec{S}$; $-\vec{E}, +\vec{B}$

C: $C(f\bar{f}) = (-1)^{l+s}$; $C_{\pi^0} = +1$; $C_\gamma = -1$ — $\vec{x}, \vec{v}, \vec{p}, \vec{L}, \vec{S}$; $-\vec{E}, -\vec{B}$

T: $\vec{x}, -\vec{v}, -\vec{p}, -\vec{L}, -\vec{S}$; $+\vec{E}, -\vec{B}$

Perturbation Theory

$H = H_0 + V(\vec{x},t)$ with $H_0|u_n\rangle = E_n|u_n\rangle$

By expanding $|\Psi(\vec{x},t)\rangle = \sum_n a_n(t)e^{-E_nt}|u_n\rangle$ we find $\frac{da_f}{dt} = -i\sum_n a_n(t)e^{i(E_f - E_i)t}V_{fn}(t)$ with

$V_{fn}(t) := \int d^3\vec{x}\, u_f^*(\vec{x})V(\vec{x},t)u_n(\vec{x})$

Born approx.: $\begin{cases} a_i(t) \approx 1 \\ a_n(t) \approx 0 \end{cases} \rightarrow a_f(+\frac{T}{2}) = -i\int_{-T/2}^{T/2}dt'\int d^3\vec{x}\, u_f^* V(\vec{x},t) u_i e^{i(E_f - E_i)t'}$

Constant $V(\vec{x},t) \equiv V(\vec{x})$: $W_{fi} = \lim_{T\to\infty}\frac{|a_f|^2}{T} = (2\pi)|V_{fi}|^2\delta(E_f - E_i)$

→ Fermi's Golden Rule: $W_{fi} = (2\pi)|V_{fi}|^2\rho(E_f)$

Elastic scattering: for $u_i(x) = \frac{1}{L^{\frac{3}{2}}}e^{i\vec{p}_i\vec{x}}$ $|\vec{p}_f| = |\vec{p}_i| =: p$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{J}\frac{dW_{fi}}{d\Omega} = \left(\frac{m}{2\pi}\right)^2\left|\int d^3\vec{x}\, V(\vec{x})e^{-i(\vec{p}_f - \vec{p}_i)\vec{x}}\right|^2$$

Rutherford $(\vec{q} = \vec{p}_f - \vec{p}_i)$: $V(\vec{x}) = \frac{ze^2}{4\pi\varepsilon_0}\left(\frac{Z}{|\vec{x}|} - \int\frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}d^3\vec{x}'\right)$ $\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2\left(\frac{ze^2}{\varepsilon_0}\right)^2\left(\frac{1}{q^4}\right)\left[Z - F(\vec{q})\right]^2$

$F(\vec{q}) = \int d^3\vec{x}\, e^{i\vec{q}\vec{x}}\rho(\vec{x})$

5. Relativistic Formulation and Kinematics

Poincaré and Lorentz group

$$x'^\mu = \overbrace{\underbrace{\Lambda^\mu{}_\nu x^\nu}_{\text{Lorentz}} + a^\mu}^{\text{Poincaré}} \quad \text{with} \quad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

inverse

Tensors: $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu \quad x_\mu \rightarrow \Lambda_\mu{}^\nu x_\nu$

$\tau = \frac{t}{\gamma} \quad V^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}) \quad p^\mu = m\gamma(c, \vec{v}) = (\frac{E}{c}, \vec{p})$

$\rightarrow p^2 = E^2 + \vec{p}^2 = m^2$

Classification of tensors:
$x^\mu x_\mu > 0 \rightarrow$ timelike
$x^\mu x_\mu = 0 \rightarrow$ lightlike
$x^\mu x_\mu < 0 \rightarrow$ spacelike

Classification of Lorentz group matrices:
$\det \Lambda = 1 \rightarrow$ proper
$\det \Lambda = -1 \rightarrow$ improper
$\Lambda^0{}_0 > 0 \rightarrow$ orthochronous
$\Lambda^0{}_0 < 0 \rightarrow$ non-orthochronous

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}}_{\text{rotation}} \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{boost}} \underbrace{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{T}$$

Lorentz-Invariant Phase Space

$A + B \rightarrow 1 + 2 + 3 + \ldots + n$

$$\boxed{\Pi_n = \int \underbrace{\prod_{i=1}^{n} \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}} (2\pi)^4 \delta^4(p_A + p_B - \sum_i p_i)}$$

this is Lorentz-invariant, which can be seen by applying a Lorentz boost or by comparing it to $\int d^4p \, \delta(p^2 - m^2)\Theta(p^0)$

Relativistic Collisions

$$\boxed{d\sigma = |M|^2 \frac{1}{F} S \left(\prod_i \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4 \delta^4\left(p_A + p_B - \sum_i p_i\right)}$$

Møller flux factor: $F = 4|\vec{p}_A E_B - \vec{p}_B E_A| = 4\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}$

2-Body CMS: $\Pi_2 = \int d\Omega \frac{1}{16\pi^2} \frac{p_f}{E_*}$

$$\rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{64\pi^2 s} \frac{p_f}{p_i} |M|^2}$$

2-Body TS: $\Pi_2 = \int d\Omega \frac{E_2 p_1}{(2\pi)^2 4E_2 (E_A + M_B - p_A \frac{E_1}{p_1}\cos\Theta)}$

$$\rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{p_1}{64\pi^2 p_A M_B (E_A + M_B - p_A \frac{E_1}{p_1}\cos\Theta)} |M|^2}$$

Relativistic decay rates

$R \rightarrow 1 + 2 + \ldots + n$

$$\boxed{d\Gamma = |M|^2 \frac{1}{2E_A} S \left(\prod_i \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4 \delta^4\left(p_A - \sum_i p_i\right)}$$

$\rightarrow \tau = \frac{1}{\Gamma} \quad \Gamma = \sum_n \Gamma_n$ ← partial decay width

lifetime

total decay width

Inertial frames in collisions

$A + B \rightarrow C + D$

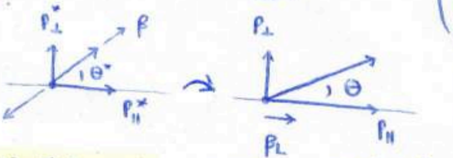
Lab system: frame in which experiment is performed
Center-of-mass system: $\vec{p}_A + \vec{p}_B = 0$
Fixed target system: $\vec{p}_B = 0$

a part of the energy represents the motion of the center-of-mass relative to the laboratory

Fixed target: $E^* = \sqrt{M_A^2 + M_B^2 + 2E_A M_B} \approx \sqrt{2E_A M_B} < E_A + M_B$
Collider: $E^* = E_A + E_B$

Example: $e^+e^- \rightarrow Z^0$
Fixed target: $E^* \geq M_Z \rightarrow E_A \geq \frac{M_Z^2}{2M_B} \approx 8 \cdot 10^6$ GeV
Collider $E^* \geq M_Z \rightarrow E_A = M_Z/2 \approx 45$ GeV
collider setup clearly favorable to reach high E^*

Transition CMS → LS:



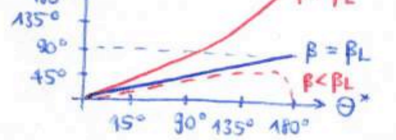
$$\begin{pmatrix} E \\ p_\parallel \\ p_\perp \end{pmatrix} = \begin{pmatrix} \gamma_L & \gamma_L\beta_L & 0 \\ \gamma_L\beta_L & \gamma_L & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E^* \\ p_\parallel^* \\ p_\perp^* \end{pmatrix}$$

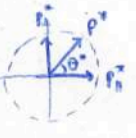
$$\tan\Theta = \frac{1}{\gamma_L}\left(\frac{\sin\Theta^*}{\frac{\beta_L}{\beta} + \cos\Theta^*}\right)$$

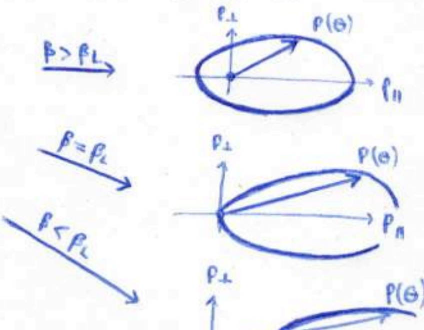
$p_\parallel^{*2} + \vec{p}_\perp^{*2} = \text{const.} \rightarrow \frac{\vec{p}_\perp^2}{a^2} + \frac{(p_\parallel - c)^2}{b^2} = 1$

c: $\beta_L\gamma_L E^*$; a^2: $(\vec{p}^*)^2$; b^2: $(\gamma_L \vec{p}^*)^2$

For $p_\perp^* = 0 \rightarrow p_\parallel = \gamma_L E^*(\beta_L \pm \beta)$







Example: $\pi^0 \rightarrow \gamma\gamma$

$\tan\Theta = \frac{\sin\Theta^*}{\gamma_L(\beta_L + \cos\Theta^*)} \quad 1 - \cos\alpha = \frac{2}{\gamma_L^2(1 - \beta_L^2\cos^2\Theta^*)}$

opening angle betw. photons

$\rightarrow \sin\left(\frac{\alpha_{min}}{2}\right) = \frac{1}{\gamma_L}$ (~15° for $E_{\pi^0} = 1$ GeV)

$\rightarrow E_{min} = E_{\pi^0}\frac{1-\beta}{2}, \; E_{max} = E_{\pi^0}\frac{1+\beta}{2}$
$(\Theta^* = 180°) \quad (\Theta^* = 0°)$

Energy distribution: $\frac{dn(E)}{dE} = \frac{dn(\cos\Theta^*)}{d\cos\Theta^*} \frac{d\cos\Theta^*}{dE} = \text{const.} \cdot \frac{2}{\beta E_{\pi^0}} \rightarrow$

2-Body CMS: $\Pi_2 = \frac{p_1}{16\pi^2 M_A} d\Omega$

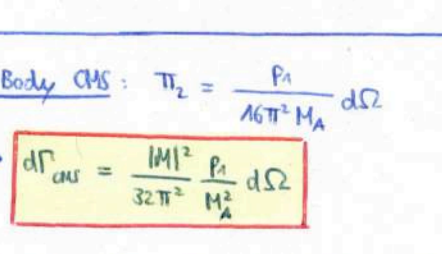
$$\rightarrow \boxed{d\Gamma_{CMS} = \frac{|M|^2}{32\pi^2} \frac{p_1}{M_A^2} d\Omega}$$

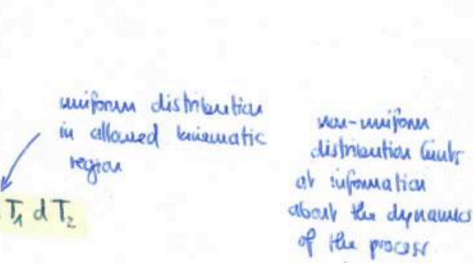
uniform distribution in allowed kinematic region

3-Body CMS: $\Pi_3 = \pi^2 dE_1 dE_2 = \pi^2 dT_1 dT_2$

non-uniform distribution hints at information about the dynamics of the process (resonances!)

Dalitz plots:





$Q = M - m_1 - m_2 - m_3 = T_1 + T_2 + T_3$

$m_{12}^2 = (p_1 + p_2)^2 = M^2 + m_3^2 - 2M(m_3 + T_3)$
$\rightarrow m_{12}^2 \leq (M - m_3)^2$

General Mechanics

Lagrangian formalism: $L(q_i, \dot{q}_i) = T - V$

$$S([q]) = \int_{t_1}^{t_2} dt\, L \quad \delta S \overset{!}{=} 0 \leftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dq_i}{dt}\right)}\right) - \frac{\partial L}{\partial q_i} \overset{!}{=} 0$$

Hamiltonian formalism: $p_i := \frac{\partial L}{\partial\left(\frac{dq_i}{dt}\right)}$

$$H(p_i, q_i, t) = p_i\left(\frac{dq_i}{dt}\right) - L(q_i, \dot{q}_i, t)$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Relativistic tensor fields

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$$

$$A^\mu(x) \rightarrow A^{\mu\prime}(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

$$F^{\mu\nu}(x) \rightarrow F^{\mu\nu\prime}(x) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}(\Lambda^{-1}x)$$

Lagrangian density in field theory

$$\mathcal{L} := \mathcal{L}(\phi, \partial_\mu\phi), \quad S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$
$$\rightarrow [\mathcal{L}] = [m^4]$$

$$\delta S \overset{!}{=} 0 \leftrightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \overset{!}{=} 0$$

(complex scalar fields:
$\varphi(x) = \phi_1(x) + i\phi_2(x) \rightarrow$ E.L. for φ, φ^*)

Space-time translation: $x^\mu \rightarrow x^{\mu\prime} = x^\mu + a^\mu$

$$\phi'(x') = \phi(x) \rightarrow \phi'(x) \approx \phi(x) - \partial_\mu\phi\, \delta a^\mu \rightarrow \delta\phi = -\delta a^\mu \partial_\mu\phi$$

$$\left.\begin{aligned} \delta\mathcal{L} &\overset{\text{E.L.}}{=} \dots = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \quad \text{in general} \\ \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta a^\mu = \partial_\mu\left(\delta^\mu_\nu \mathcal{L}\right)\delta a^\nu \quad \text{here} \end{aligned}\right\} \quad T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial^\nu\phi) - \eta^{\mu\nu}\mathcal{L}$$

is a conserved current

($\partial_\mu T^{\mu\nu} = 0 \rightarrow \int d^3\vec{x}\, T^{0\nu}$ cons.)

$$\mathcal{H} := T^{00} = \overbrace{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}}^{\pi}\partial^0\phi - \mathcal{L}$$

$$\mathcal{P}^i := T^{0i} = \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}}_{\pi}\partial^i\phi$$

6. The Lagrangian Formalism

General Noether currents

Transformation: $x^\mu \rightarrow x^{\mu\prime} = x^\mu + \delta x^\mu$
$\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta_\phi\phi(x)$

$$\delta S = \int \delta(d^4x)\mathcal{L} + \int d^4x\, \delta\mathcal{L} \cong \int d^4x\, (1 + \partial_\nu\delta x^\nu)\mathcal{L} +$$
$$\int d^4x \left((\partial_\nu\mathcal{L})\delta x^\nu + \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta_\phi\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\phi\phi\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta_\phi\phi\right)$$
$$= \dots = \underbrace{\int_{\partial\Omega}\left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta_\phi\phi - T_\nu{}^\mu\delta x^\nu\right]d^3S_\mu}_{\rightarrow \text{ conserved current}} + \underbrace{\int_\Omega d^4x\, \delta_\phi\phi\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]}_{\rightarrow \text{ Euler-Lagrange eqs.}}$$

Conserved current: $\delta_\phi\phi = \Phi_s\,\delta\alpha^s$
$\delta x^\nu = X^\nu_s\,\delta\alpha^s$

$$\rightarrow \boxed{J^\mu_s = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\Phi_s - T^\mu{}_\nu X^\nu_s} \qquad \left(J^\mu_s = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\Phi_s + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\right)\Phi^*_s - T^\mu{}_\nu X^\nu_s\right)$$

satisfies $\partial_\mu J^\mu_s = 0$

Conserved charge: $\boxed{Q_s = \int d^3\vec{x}\, J^0_s}$

satisfies $\frac{d}{dt}Q_s = 0$

Examples: (translation) $x^\mu \rightarrow x^{\mu\prime} = x^\mu + a^\mu$

$\rightarrow J^\mu_s = -T^\mu{}_s$

(global gauge) $\varphi(x) \rightarrow \varphi'(x) = e^{-i\alpha}\varphi(x)$

$\rightarrow J^\mu_s = -i\left(\varphi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \varphi^*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\right)$

8. Free Fermion Dirac Fields (1/2)

Dirac equation

Postulate: $\boxed{H_D = \vec\alpha\cdot\vec p + \beta m}$ with $E^2 = (\vec\alpha\vec p + \beta m)^2 = \vec p^{\,2} + m^2$

$$\rightarrow \begin{cases} \alpha_i^2 = \beta^2 = \mathbb{1} & \alpha_i^\dagger = \alpha_i \\ \{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0 & \beta^\dagger = \beta \end{cases} \quad \mathrm{tr}(\alpha_i) = \mathrm{tr}(\beta) = 0$$

Pauli-Dirac representation: $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

Chiral/Weyl representation: $\beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ $\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$

$\psi_{PD} = \mathcal{R}\psi_{\text{chiral}}\mathcal{R}^{-1}$, with $\mathcal{R} = \frac{1}{\sqrt2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}$, $\mathcal{R}^{-1} = \frac{1}{\sqrt2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$

Covariant form: $\gamma^0 = \beta$, $\gamma^k = \beta\alpha_k$ $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}}$

$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}$

Dirac equation: $\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$

$\rightarrow (i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = -(\partial^\mu\partial_\mu + m^2)\psi \overset{!}{=} 0$

$\rightarrow$ each spinor component solves the K.G. equation

Current density: defining $\bar\psi = \psi^\dagger\gamma^0$, one finds that $\boxed{J^\mu = \bar\psi\gamma^\mu\psi}$ is conserved

$\rightarrow \rho = \psi^\dagger\psi \geq 0$

Dirac particles and spin

$$\left.\begin{array}{l} [\vec L, H_{\text{Dirac}}] = i\vec\alpha\times\vec p \neq 0 \\ \vec S := \frac{1}{2}\begin{pmatrix} \vec\sigma & 0 \\ 0 & \vec\sigma \end{pmatrix} \\ [\vec S, H_{\text{Dirac}}] = -i\vec\alpha\times\vec p \neq 0 \end{array}\right\} \boxed{[\vec L + \vec S, H_{\text{Dirac}}] = 0}$$

For $\vec p = (0, 0, \pm p)$: $\begin{cases} \psi^{(1)}, \psi^{(3)} \leftrightarrow |\uparrow\rangle \\ \psi^{(2)}, \psi^{(4)} \leftrightarrow |\downarrow\rangle \end{cases}$

Anti-particle spinors

introduce $\boxed{\phi^{(i)}(x^\mu) = v^{(i)}(E,\vec p)e^{ipx}} = u^{(i)}(-E,-\vec p)e^{-i(-p)x}$

$\rightarrow \boxed{v^{(1,2)}(p) \leftrightarrow u^{(4,3)}(-p)}$

Complete set: $u^{(1,2)}(E,\vec p) = \sqrt{E+m}\begin{pmatrix} u_A^{(1,2)} \\ \left(\frac{\vec\sigma\cdot\vec p}{E+m}\right)u_A^{(1,2)} \end{pmatrix}$ $v^{(1,2)}(E,\vec p) = \sqrt{E+m}\begin{pmatrix} \left(\frac{\vec\sigma\vec p}{E+m}\right)u_A^{(2,1)} \\ u_A^{(2,1)} \end{pmatrix}$

$\psi^{(i)} = u^{(i)}e^{-ipx}$ | $\phi^{(i)} = v^{(i)}e^{ipx}$

$(p^\mu\gamma_\mu - m)\psi = 0$ | $(p^\mu\gamma_\mu + m)\phi = 0$

$\bar u^{(r)}u^{(s)} = 2m\delta_{r,s}$ | $\bar v^{(r)}v^{(s)} = -2m\delta_{r,s}$

$\boxed{\sum_{s=1,2} u^{(s)}(p)\bar u^{(s)}(\vec p) = \gamma^\mu p_\mu + \mathbb{1}m}$ $\boxed{\sum_{s=1,2} v^{(s)}(p)\bar v^{(s)}(p) = \gamma^\mu p_\mu - \mathbb{1}m}$

Dirac spinors (eigenstates)

Ansatz: $\boxed{\psi^{(i)}(x^\mu) = u^{(i)}(E,\vec p)e^{-ipx}}$

$$E>0: \psi^{(1)} = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}e^{-ipx} \quad \psi^{(2)} = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}e^{-ipx}$$

$$E<0: \psi^{(3)} = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}e^{-ipx} \quad \psi^{(4)} = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}e^{-ipx}$$

$N = \sqrt{|E|+m}$ to have $\rho = 2|E|$

Completeness relation: $u^{(r)\dagger}u^{(s)} = 2|E|\delta_{r,s}$

$\bar u^{(r)}u^{(s)} = \frac{|E|}{E}2m\delta_{r,s}$

$\rightarrow \boxed{\sum_{s=1,2} u^{(s)}(\vec p)\bar u^{(s)}(\vec p) = \gamma^\mu p_\mu + m\mathbb{1}}$

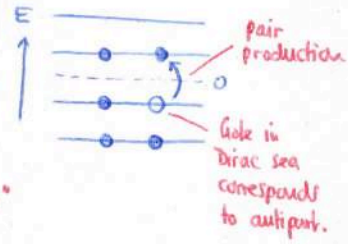
$\Lambda_+ = \frac{\not p + m\mathbb{1}}{2m}$, $\Lambda_- = \frac{-\not p + m\mathbb{1}}{2m}$ project out positive/negative energy solutions

Antiparticles

How can the negative energy solutions be explained?

Dirac's Sea Picture:

all E<0 states are populated → Fermi exclusion prohibits particles to "fall down"



Discovery of positron:

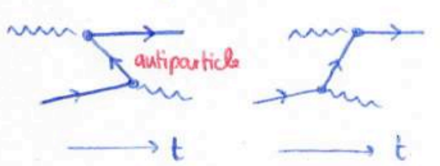
Tracks on Cloud chamber pictures
→ supersaturated vapor
→ bubbles form along the trajectories of ionizing particles

"Particle identification" via $p[\text{GeV}] = R[\text{m}]B[\text{T}]\cdot 0{,}3$ and $\langle\frac{dE}{dx}\rangle \propto \frac{1}{v^2}$

Stückelberg-Feynman:

negative energy propagating backward in time
⟷ positive energy antiparticle propagating forward in time

$e^{-i(-E)(-t)} = e^{-i(+E)(+t)}$



8. Free Fermion Dirac Fields (2/2)

Helicity

$$\boxed{h = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} = \begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix}}$$

$[h, H_{Dirac}] = 0$

$\Rightarrow \rightarrow p$ $+1$

$\Leftarrow \rightarrow p$ -1

<u>Eigenstates</u>: $u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ s e^{i\varphi} \\ \alpha c \\ \alpha s e^{i\varphi} \end{pmatrix}$ $u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ c e^{i\varphi} \\ \alpha s \\ -\alpha c e^{i\varphi} \end{pmatrix}$

$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \alpha s \\ -\alpha c e^{i\varphi} \\ -s \\ c e^{i\varphi} \end{pmatrix}$ $v_\downarrow = \sqrt{E+m}\begin{pmatrix} \alpha c \\ \alpha s e^{i\varphi} \\ c \\ s e^{i\varphi} \end{pmatrix}$

with $\alpha = \frac{p}{E+m}$, $c = \cos\left(\frac{\theta}{2}\right)$, $s = \sin\left(\frac{\theta}{2}\right)$

Chirality and helicity

chirality = eigenstates of γ^5 $\neq$ helicity = projection of spin operator onto the direction of motion

$u_\uparrow = \frac{1}{2}(1-\alpha)u_L + \frac{1}{2}(1+\alpha)u_R$ $\frac{p}{E+m}$

→ in the highly relativistic limit: $(E \gg m)$

$$\boxed{\begin{matrix} u_\uparrow \leftrightarrow u_R & v_\uparrow \leftrightarrow v_L \\ u_\downarrow \leftrightarrow u_L & v_\downarrow \leftrightarrow v_R \end{matrix}}$$

⚠ here: $\gamma^5 v_L = -v_L$, but sometimes they are defined as $\gamma^5 v_L = v_L$, s.t. $v_L \leftrightarrow v_\downarrow$

Dirac Lagrangian

$S^\dagger(\Lambda)S(\Lambda) = \gamma^0 S(\Lambda)^{-1}\gamma^0 S(\Lambda)$

→ $\bar{\psi}(x)\psi(x)$ is a Lorentz scalar

→ $\bar{\psi}(x)\gamma^\mu\psi$ is a Lorentz 4-vector

→ $\boxed{\mathcal{L}_{Dirac} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi}$ has all the right properties

<u>Generally</u>: $\bar{\psi}\psi \rightarrow$ scalar, $\bar{\psi}\gamma^5\psi \rightarrow$ pseudo-scalar

$\bar{\psi}\gamma^\mu\psi \rightarrow$ vector, $i\bar{\psi}\gamma^\mu\gamma^5\psi \rightarrow$ axial-vector

$\frac{i}{2}[\gamma^\mu,\gamma^\nu]$ $i\bar{\psi}\sigma^{\mu\nu}\psi \rightarrow$ tensor

→ "mix and match" to get the desired properties for the Lagrangian

Lorentz transformation of spinors

$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x)$ with $\boxed{S(\Lambda) = \exp\left(\frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu}\right)}$

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ if we set $\Omega_{\mu\nu} = -\omega_{\mu\nu}$

<u>Pauli-Dirac representation</u>: $S^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ (boosts)

$S^{ij} = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$ (rotations)

→ $S(\Lambda)_{PD} = \sqrt{\frac{E-m}{2m}}\begin{pmatrix} \mathbb{1} & \frac{\vec{\sigma}\vec{p}}{E+m} \\ \frac{\vec{\sigma}\vec{p}}{E+m} & \mathbb{1} \end{pmatrix}$ boost to the reference frame of a particle having $p^\mu = (E, \vec{p})$

<u>Weyl representation</u>: $S^{0i} = \frac{i}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$ (boosts)

$S^{ij} = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$ (rotations)

→ $S(\Lambda)_{chiral} = \begin{pmatrix} e^{-\frac{1}{2}\vec{\sigma}\vec{\phi}} & 0 \\ 0 & e^{\frac{1}{2}\vec{\sigma}\vec{\phi}} \end{pmatrix}$ boost in the direction $\vec{\phi} = (\phi_x, \phi_y, \phi_z)$

<u>Chirality</u>: in the Weyl representation, $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi_{chiral}$

→ ψ_L, ψ_R are <u>not</u> mixed under Lorentz transformations

→ this can be generalized by defining $\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$ with $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^2 = 1$, $(\gamma^5)^\dagger = \gamma^5$, $[S^{\mu\nu}, \gamma^5] = 0$

$\gamma^5_{PD} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^5_{Weyl} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

→ $\boxed{P_L = \frac{1}{2}(1-\gamma^5)}$ and $\boxed{P_R = \frac{1}{2}(1+\gamma^5)}$ project out the left-handed / right-handed chirality components

Discrete symmetries

<u>Parity transformation</u>: $P\psi(t,\vec{x})P = \eta_P\gamma^0\psi(t,-\vec{x})$

<u>Time reversal</u>: $T\psi(t,\vec{x})T = -\eta_T\gamma^1\gamma^3\psi(-t,\vec{x})$

<u>Charge conjugation</u>: $\psi \rightarrow \psi_c = -i\eta_c\gamma^2\psi^*(t,\vec{x})$

<u>CPT theorem</u>: $S(CPT)\psi(x) = -\eta_{CPT}\gamma^5\psi^*(-x)$

→ $m = \bar{m}$, $\tau = \bar{\tau}$, $\mu = -\bar{\mu}$

Second quantization

$$\boxed{\begin{aligned} \psi &= \int\frac{d^3\vec{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(a_s(\vec{p})u^{(s)}(\vec{p})e^{-ipx} + b_s^\dagger(\vec{p})v^{(s)}(\vec{p})e^{ipx}\right) \\ \bar{\psi} &= \int\frac{d^3\vec{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(a_s^\dagger(\vec{p})\bar{u}^{(s)}(\vec{p})e^{ipx} + b_s(\vec{p})\bar{v}^{(s)}(\vec{p})e^{-ipx}\right) \end{aligned}}$$

<u>Equal-time "commutation"</u>: $\{\psi_i(\vec{x},t), \psi_j^\dagger(\vec{x}',t)\} = \delta^{(3)}(\vec{x}-\vec{x}')\delta_{ij}$

$\{\psi_i(\vec{x},t), \psi_j(\vec{x}',t)\} = \{\psi_i^\dagger(\vec{x},t), \psi_j^\dagger(\vec{x}',t)\} = 0$

<u>Anti-commutation</u>: $\boxed{\{a_r(\vec{p}), a_s^\dagger(\vec{p}')\} = \{b_r(\vec{p}), b_s^\dagger(\vec{p}')\} = \delta^{(3)}(\vec{p}-\vec{p}')\delta_{r,s}}$

$\{a_r(\vec{p}), a_s(\vec{p}')\} = \{a_r^\dagger(\vec{p}), a_s^\dagger(\vec{p}')\} = 0$

<u>Hamiltonian</u>: $H = \int d^3\vec{p}\sum_{s=1,2}E_p\left(a_s^\dagger(\vec{p})a_s(\vec{p}) + b_s^\dagger(\vec{p})b_s(\vec{p})\right)$

$\vec{P} = \int d^3\vec{p}\sum_{s=1,2}\vec{p}\left(a_s^\dagger(\vec{p})a_s(\vec{p}) + b_s^\dagger(\vec{p})b_s(\vec{p})\right)$

Direct interaction terms

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad H = H_0 + H_{int}$ ← direct interact. term

Direct or non-derivative coupling: $\mathcal{L}_{int}$ does only depend on ϕ, not $\partial_\mu \phi$

Toy Model

σ^0 of mass $M \longrightarrow \sigma(x)$

$\pi^\pm$ of mass $m \rightarrow \varphi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

π^0 of mass $m \longrightarrow \phi(x)$

Possible interactions: $g\phi^4, g'\sigma\phi^3, \dots, \lambda\phi^3, \lambda'\sigma\phi^2, \dots$

→ additional constraints due to parity: $\phi(x) \rightarrow -\phi(x)$

→ there needs to be an even number of $\phi(x)$ factors

Neutral decay: $\sigma \rightarrow \pi^0\pi^0$: $H_{int} = \lambda\sigma\phi^2(x)$

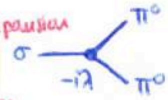
$$\rightarrow \boxed{S_1 = (-i)\langle q_1 q_2|\int dx_1^4\, \lambda\sigma(x_1)\phi^2(x_1)\,|k\rangle}$$

with $|q_1 q_2\rangle = \sqrt{(2\pi)^3 2e(\vec{q}_1)}\sqrt{(2\pi)^3 2e(\vec{q}_2)}\; a^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle$

$|k\rangle = \sqrt{(2\pi)^3 2e(\vec{k})}\; c^\dagger(\vec{k})|0\rangle$

plugging in Fourier mode expansion

$$\rightarrow S_1 = \underbrace{(-2i\lambda)}_{iM_1}(2\pi)^4\delta^4(q_1+q_2-k)$$



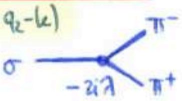
factor of 2 since particles are indistinguishable → two diagrams

$$\rightarrow d\Gamma = \frac{\lambda^2}{32\pi^2 M}\sqrt{1-\left(\frac{2m}{M}\right)^2}\, d\Omega$$

Charged decay: $\sigma \rightarrow \pi^+\pi^-$: $H_{int} = \lambda\sigma(\phi_1^2+\phi_2^2) = 2\lambda\sigma(\varphi^\dagger\varphi)$

$$\rightarrow \boxed{S_1 = -i\langle q_1 q_2|\int dx_1^4\, 2\lambda\sigma(x_1)\varphi^\dagger(x_1)\varphi(x_1)|k\rangle}$$

$$= \dots = \underbrace{(-2i\lambda)}_{iM_1}(2\pi)^4\delta^4(q_1+q_2-k)$$

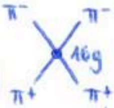


$$\rightarrow d\Gamma = \frac{\lambda^2}{16\pi^2 M}\sqrt{1-\left(\frac{2m}{M}\right)^2}\, d\Omega$$

Charged scattering: $\pi^+\pi^- \rightarrow \pi^+\pi^-$: $H_{int} = 4g(\varphi^\dagger\varphi)^2$

$$\rightarrow S_1 = -i\langle q_3 q_4|\int dx_1\, 4g(\varphi^\dagger(x_1)\varphi(x_1))^2|q_1 q_2\rangle$$

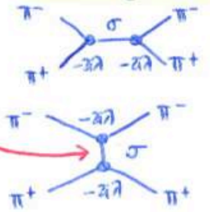
$$= \dots = \underbrace{(-16ig)}_{iM_1}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)$$



$$\|$$

$H_{int} = 2\lambda\sigma(\varphi^\dagger\varphi)$

$$\rightarrow \boxed{S_2 = \frac{(-2\lambda i)^2}{2}\langle q_3 q_4|\int dx_1^4\int dx_2^4\, T\left[\sigma(x_1)(\varphi^\dagger(x_1)\varphi(x_1))\,\sigma(x_2)(\varphi^\dagger(x_2)\varphi(x_2))\right]|q_1 q_2\rangle} = \dots = (-2\lambda i)^2\left[\tilde{D}_F(q_1+q_2) + \tilde{D}_F(q_1-q_3)\right]$$



virtual exchange boson

↳ action at a distance

Time Evolution Operator

Heisenberg picture: $\phi(x^\mu) = e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)}$

Interaction picture: $\phi_I(x^\mu) = e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}$ ← operator evolves with the free Hamiltonian

$$\longrightarrow \phi(x^\mu) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})U(t,t_0) \quad \text{w/} \quad \boxed{U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}$$

$$\longrightarrow i\partial_t U(t,t_0) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}U(t,t_0)$$

$$\longrightarrow U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, H_{int}(t')\cdot U(t',t_0) \xrightarrow{\text{Dyson}} \boxed{U(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt'\, H_{int}(t')\right)\right]}$$

S-matrix: $S := \lim_{T\to\infty} U\left(\frac{T}{2}, -\frac{T}{2}\right) = U(+\infty, -\infty) =$

$$1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int dx_1^4 \dots \int dx_n^4\; T\left[\mathcal{H}_{int}(x_1)\cdots\mathcal{H}_{int}(x_n)\right]$$

$$S^\dagger S = 1 \rightarrow \text{writing } S = 1 + iM \longrightarrow \boxed{M^\dagger M = 2\,\mathrm{Im}\{M\}}$$

Feynman Propagators

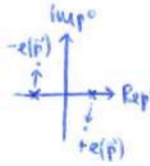
interaction at a distance → exchange of a field quantum

$\phi(x^\mu)|0\rangle = \int\frac{d^3\vec{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2e(\vec{p})}}a^\dagger(\vec{p})e^{ipx}|0\rangle$ → creates a particle at point x

$$\boxed{D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2e(\vec{p})}e^{-ip(x-y)}}$$ creates at y and annihil. at x.

$$\boxed{D_F(x-y) = \langle 0|T[\phi(x)\phi(y)]|0\rangle} = \Theta(x^0-y^0)D(x-y) + \Theta(y^0-x^0)D(y-x)$$

$$= i\lim_{\varepsilon\to 0^+}\int\frac{dp^4}{(2\pi)^4}\frac{1}{2e(\vec{p})}\frac{e^{-ip(x-y)}}{(p^0-e(\vec{p})+i\varepsilon)} + i\lim_{\varepsilon\to 0^+}\int\frac{dp^4}{(2\pi)^4}\frac{1}{2e(\vec{p})}\frac{e^{-ip(x-y)}}{-p^0-e(\vec{p})+i\varepsilon}$$



9. Interacting Fields and Propagator Theory

$$= \boxed{i\int\frac{dp^4}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}} \longrightarrow \boxed{\tilde{D}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}}$$

Green's function: $(\partial^\mu\partial_\mu + m^2)\underbrace{i\int\frac{dp^4}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2}}_{D_F(x-y)} = \dots = \boxed{-i\delta^4(x-y)}$

Complex field: $D^q(x-y) = \langle 0|\varphi(x)\varphi^\dagger(y)|0\rangle$

$$\boxed{D_F^q(x-y) = \langle 0|T[\varphi(x)\varphi^\dagger(y)]|0\rangle = \int\frac{dp^4}{(2\pi)^4}e^{-ip(x-y)}\overbrace{\frac{i}{p^2+m^2+i\varepsilon}}^{\tilde{D}_F(p)}}$$

since $(-p^2+m^2)\tilde{D}_F(p) = -i$ to satisfy the Green's fct. condition

Dirac Field: $S_{\alpha\beta}(x-y) := \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$

$S_F(x-y) := \langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle$

→ both time-orderings are considered

Using $(i\gamma^\mu\partial_\mu - m)S_F(x-y) = i\delta^4(x-y)$:

$$\boxed{S_F(x-y). = i\int\frac{dp^4}{(2\pi)^4}e^{-ip(x-y)}\frac{p^\mu\gamma_\mu + m1}{(p^2-m^2+i\varepsilon)}} \longrightarrow \tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2} = \frac{i}{p^2-m^2}\left(\sum_{s=1,2}u^{(s)}\bar{u}^{(s)}\right)$$

→ numerator is the sum over all internal degrees of freedom

Wick's theorem

Defining $\phi(x) = \overbrace{\phi^+(x)}^{\sim a} + \overbrace{\phi^-(x)}^{\sim a^\dagger} \longrightarrow T[\phi(x)\phi(y)] = N[\phi(x)\phi(y)] + \underbrace{\begin{cases}[\phi^+(x),\phi^-(y)], & x^0>y^0\\ [\phi^+(y),\phi^-(x)], & y^0>x^0\end{cases}}_{\boxed{\phi(x)\phi(y)\text{ contracted}}}$

$$\longrightarrow \boxed{\langle 0|\underbrace{\phi(x)\phi(y)}|0\rangle = D_F(x-y)}$$

Wick's theorem: $\boxed{T[\phi(x_1)\cdots\phi(x_n)] = N[\phi(x_1)\cdots\phi(x_n) + \text{all possible contractions}]}$

Complex scalar field: $\langle 0|\overbrace{\varphi(x)\varphi^\dagger(y)}|0\rangle = D_F^q(x-y)$

Dirac field: $\langle 0|\overbrace{\psi(x)\bar{\psi}(y)}|0\rangle = \begin{cases}\{\psi^+(x),\bar{\psi}^-(y)\}, & x^0>y^0\\ ⊖\{\bar{\psi}^+(y),\psi^-(x)\}, & y^0>x^0\end{cases} = S_F(x-y)$

10. Quantum Electro-dynamics (1/2)

Maxwell theory

$\vec\nabla\vec E=\rho$ $\quad\vec\nabla\times\vec B-\dot{E}=\vec J$ $\quad\vec B=\vec\nabla\times\vec A$

$\vec\nabla\vec B=0$ $\quad\vec\nabla\times\vec E+\dot{B}=0$ $\quad\vec E=-\vec\nabla\phi-\dot{\vec A}$

→ Ehrenberg - Siday - Aharonov - Bohm effect
→ potentials are physical

Covariant formulation: $\boxed{F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu}$ with $F^{\mu\nu}=\begin{pmatrix}0&-E_x&-E_y&-E_z\\ &0&-B_z&B_y\\ &&0&-B_x\\ &&&0\end{pmatrix}$

and $A^\mu=(\phi,\vec A)\longrightarrow\boxed{\partial_\mu F^{\mu\nu}=J^\nu}$, $J^\nu=(\rho,\vec J)$

$$\boxed{\mathcal{L}_{\text{Maxwell}}=\underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\mathcal{L}_{\text{Proca}}}-\underbrace{J_\mu A^\mu}_{\mathcal{L}_{\text{int}}}}$$ yields $\partial_\mu F^{\mu\nu}=J^\nu$

Gauge freedom

$\boxed{A^\mu\to A^\mu+\partial^\mu\lambda}$ leaves $F^{\mu\nu}$ invariant
→ redundancy

Lorenz gauge: $\boxed{\partial_\mu A^\mu=0}$ → reduces d.o.f. by one, $\partial_\mu\partial^\mu\lambda=0$ remains
(Landau gauge)

$\longrightarrow\partial_\mu F^{\mu\nu}=\partial_\mu\partial^\mu A^\nu=J^\nu$

Coulomb gauge: $\boxed{\vec\nabla\vec A=0}$ → together with $\partial_\mu A^\mu=0$ follows $A^0=0$

Second Quantization of EM field

$$\boxed{A^\mu(x)=\int\frac{d\vec k^3}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1,2}\left(\varepsilon^\mu_\lambda(\vec k)a_\lambda(\vec k)e^{-ikx}+\varepsilon^{\mu*}_\lambda(\vec k)a^\dagger_\lambda(\vec k)e^{ikx}\right)}$$

with $\boxed{[a_\lambda(\vec k),a^\dagger_{\lambda'}(\vec k')]=\delta^\lambda_{\lambda'}\,\delta^3(\vec k-\vec k')}$
(all others zero)

$\rightarrow H=\int d\vec k^3\,\omega_k\sum_{\lambda=1,2}(a^\dagger_\lambda(\vec k)a_\lambda(\vec k))$

Photon propagator: $G_F^{\mu\nu}(x-y)=\langle0|T[A^\mu(x)A^\nu(y)]|0\rangle$
$(\partial_\mu\partial^\mu\eta_{\alpha\nu}-\partial_\nu\partial_\alpha)G_F^{\alpha\nu}(x-y)=i\,\delta^4(x-y)$

$\longrightarrow\boxed{\tilde G_F^{\mu\nu}=\frac{i}{k^2}C^{\mu\nu}(k)}$ w/ $C^{\mu\nu}(k)=-g^{\mu\nu}+k^\mu C^\nu(k)+k^\nu C^\mu(k)$

and $C^\mu(k)=\frac{(k^0,-\vec k)}{2\vec k^2}$ (in Coulomb gauge)

gauge dependent

Generally: $C^\mu(k)=\xi\frac{k^\mu}{2k^2}\rightarrow\boxed{C^{\mu\nu}(k)=-g^{\mu\nu}+\xi\frac{k^\mu k^\nu}{k^2+i\varepsilon}}$

This result is found for $$\boxed{\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2(1-\xi)}\underbrace{(\partial_\mu A^\mu)^2}}$$

vanishes in Landau gauge

$\xi=1\leftrightarrow$ Landau gauge
$\xi=0\leftrightarrow$ Feynman gauge

propagator is gauge dep.

$$\rightarrow\boxed{G_F^{\mu\nu}(x-y)=\int\frac{dk^4}{(2\pi)^4}e^{-ik(x-y)}\left(\frac{-ig^{\mu\nu}}{k^2+i\varepsilon}\right)}$$ (Feynman gauge)

Completeness relation: $\sum_\lambda\varepsilon^{\mu,*}_{(\lambda)}\varepsilon^\nu_{(\lambda)}\leftrightarrow-g^{\mu\nu}+\xi\frac{k^\mu k^\nu}{k^2}$

since we expect the numerator of the propagator to be the sum over all internal degrees of freedom

Massive vector field: $(\partial^\mu\partial_\mu+m^2)B^\nu-\partial^\nu\partial_\mu B^\mu=J^\nu$
leads to $\partial_\nu B^\nu=0$ (free field), but there is <u>no</u> gauge freedom: $B^\nu\rightarrow B^\nu+\partial^\nu\lambda$ does not leave the e.o.m. invariant!

→ massive vector particles: three polarizations
massless vector particles: two polarizations

Massive vector propagator: $G_F^{\mu\nu}(x-y)=\langle0|T[B^\mu(x)B^\nu(y)]|0\rangle$

$\rightarrow((\partial_\alpha\partial^\alpha+M^2)\eta_{\mu\nu}-\partial_\mu\partial_\nu)G_F^{\mu\nu}(x-y)=i\delta^4(x-y)$

$$\rightarrow\boxed{\tilde G_F^{\mu\nu}(k)=\frac{i\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{k^2}\right)}{k^2-M^2+i\varepsilon}}$$

Photon polarization states: Photons are spin-1 particles
→ it should in principle have $m_s=-1,0,1$, but $m_s=0$ does not exist for massless particles.
longitudinal spin state

How can a photon having <u>two</u> spin degrees of freedom be described by A^μ having <u>four</u> degrees of freedom? → gauge freedom!

Lorenz gauge $(J^\nu=0)$: $\partial_\mu\partial^\mu A^\nu=0\rightarrow\boxed{A^\mu(x)=\varepsilon^\mu(k)e^{-ikx}}$
with $\partial_\mu A^\mu(x)=0\leftrightarrow\boxed{k_\mu\varepsilon^\mu(k)=0}$

The transformation $\partial_\mu\partial^\mu\lambda=0\rightarrow\lambda(x)=-iae^{-ikx}$
$\rightarrow\varepsilon^\mu(k)\rightarrow\varepsilon^\mu(k)+ak^\mu$ remains $\longrightarrow$ choose $\boxed{\varepsilon^0=0}$
$\rightarrow\vec\varepsilon\cdot\vec k=0\longrightarrow$ setting $\vec k\parallel\vec e_z$ we get the physical polarizations

$\varepsilon^\mu_{(1)}=(0,1,0,0)\quad\varepsilon^\mu_{(2)}=(0,0,1,0)$

$\varepsilon^\mu_{(+)}=-\frac{1}{\sqrt2}(0,1,i,0)\quad\varepsilon^\mu_{(-)}=\frac{1}{\sqrt2}(0,1,-i,0)$

with $\varepsilon^{*\mu}_\lambda\varepsilon_{\mu,\lambda'}=-\delta_{\lambda\lambda'}$

QED Lagrangian

Using the minimal substitution $p^\mu\rightarrow p^\mu-eA^\mu$ $(i\partial^\mu\rightarrow i\partial^\mu-eA^\mu)$ for the free Lagrangian, one obtains

$$\boxed{\mathcal{L}_{QED}=\bar\psi[i\gamma^\mu\partial_\mu-m]\psi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-e\bar\psi\gamma^\mu\psi A_\mu}$$

It is Lorentz-invariant and also gauge invariant given

$A_\mu\rightarrow A_\mu+\partial_\mu\lambda$
$\psi\rightarrow\psi e^{-i\lambda(x)e}$
$\bar\psi\rightarrow\bar\psi e^{i\lambda(x)e}$

$\boxed{D_\mu:=\partial_\mu+ieA_\mu}$

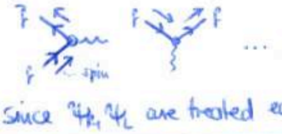
$\rightarrow\mathcal{L}_{QED}=\bar\psi[i\gamma^\mu D_\mu-m]\psi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

$D_\mu\psi\rightarrow(\partial_\mu+ieA_\mu+ie\partial_\mu\lambda)\psi e^{-i\lambda(x)}=e^{-i\lambda(x)}D_\mu\psi$

Chiral structure

$\boxed{\bar\psi\gamma^\mu\psi}=(\bar\psi_R+\bar\psi_L)\gamma^\mu(\psi_R+\psi_L)=\boxed{\bar\psi_R\gamma^\mu\psi_R+\bar\psi_L\gamma^\mu\psi_L}$, since

$\bar\psi_R\gamma^\mu\psi_L=(P_R\psi)^\dagger\gamma^0\gamma^\mu\psi_L=\psi^\dagger P_R\gamma^0\gamma^\mu\psi_L=\bar\psi P_L\gamma^\mu\psi_L$
$=\bar\psi\gamma^\mu P_R\psi_L=0\longrightarrow$ chirality is conserved at the QED vertex in the massless case!



Since ψ_R,ψ_L are treated equally, parity is conserved in QED!

Gordon identity

$$\bar u(p')\gamma^\mu u(p)=\frac{1}{2m}\bar u(p')\Big[\overbrace{(p'+p)^\mu}^{\text{electric term}}+\overbrace{i\sigma^{\mu\nu}(p'-p)_\nu}^{\text{interaction of magnetic moment → magnetic term}}\Big]u(p)$$

$\sigma^{\mu\nu}$: $\frac{i}{2}[\gamma^\mu,\gamma^\nu]$

10. Quantum Electrodynamics (2/2)

Towards the Feynman Rules

For QED : $\mathcal{H}_{int} = (e\bar{\psi}\gamma^\mu\psi)A_\mu$

$$\rightarrow S_1 = \frac{(-ie)}{1}\int dx_1^4\ \bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)$$

$$S_2 = \frac{(-ie)^2}{2}\iint dx_1^4\,dx_2^4\ T\left[\bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)\bar{\psi}(x_2)\gamma^\nu\psi(x_2)A_\mu(x_2)\right]$$

→ initial states : $|p^\mu,s\rangle = \sqrt{(2\pi)^3 2e(\vec{p})}\ a_s^\dagger(\vec{p})|0\rangle$ $|p^\mu,s\rangle = \sqrt{(2\pi)^3 2e(\vec{p})}\ b_s^\dagger(\vec{p})|0\rangle$ $|k^\mu,\varepsilon^\mu\rangle = \sqrt{(2\pi)^3 2\omega(\vec{k})}\ a_\varepsilon^\dagger(\vec{k})|0\rangle$

→ The S-matrix elements can be calculated using Wick's theorem with

$$\underbrace{\psi(x_1)\bar{\psi}(x_2)} \rightarrow \frac{i(\not{q}+m)}{q^2-m^2+i\varepsilon} \qquad \underbrace{A^\mu(x)A^\nu(y)} \rightarrow \frac{-i\,g^{\mu\nu}}{q^2+i\varepsilon}$$

the photon propagator seems to couple with four degrees of freedom to the vertices, however, gauge fixing will reduce the degrees of freedom of q^μ from four to three

→ polarizations : fermions : 2, real photons : 2, off-shell photons : 3

Real/virtual photons, Ward identity

$$\left.\begin{aligned}\mathcal{M}(X\rightarrow Y+\gamma) &= \varepsilon_\mu^*(k)M^\mu \\ \mathcal{M}(X+\gamma\rightarrow Y) &= \varepsilon_\mu(k)M^\mu\end{aligned}\right\}$$

since we have the gauge freedom $\varepsilon^\mu \rightarrow \varepsilon^\mu + a k^\mu$

→ $\boxed{k_\mu M^\mu = 0}$ for gauge invariance

(only holds for sum of all diagrams at a certain order, not for individual diagrams)

Consider now $|\varepsilon_\mu(k)M^\mu|^2 = \varepsilon^{*,\alpha}(k)\varepsilon^\beta(k)\mathcal{M}_\alpha^*\mathcal{M}_\beta$ with

$k^\mu = (\omega,0,0,|\vec{k}|)$ → Ward identity: $\omega\mathcal{M}_0 - |\vec{k}|\mathcal{M}_3 = 0$

→ for a real photon : $\mathcal{M}_0 = \mathcal{M}_3$

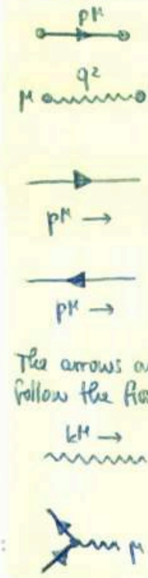
these are just the physical pol. we've seen before

$$\rightarrow \sum_{\lambda=1}^{2} \overbrace{\varepsilon_{\lambda}^{*,\alpha}(k)\varepsilon_\lambda^\beta(k)}\mathcal{M}_\alpha^*\mathcal{M}_\beta = |\mathcal{M}_1|^2+|\mathcal{M}_2|^2 = -g^{\alpha\beta}\mathcal{M}_\alpha^*\mathcal{M}_\beta$$

$$\rightarrow \boxed{\sum_{\lambda=1}^{②} \varepsilon_\lambda^{*,\mu}(k)\varepsilon_\lambda^\nu(k) \leftrightarrow -g^{\mu\nu}}$$

completeness relation for a real photon

Feynman Rules for QED



▶ R1 : $\dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$

▶ R2 : $\dfrac{-ig^{\mu\nu}}{q^2+i\varepsilon}$ (Feynman gauge)

▶ R3 : $\begin{cases} u^{(s)}(\vec{p}) \text{ - incoming} \\ \bar{u}^{(s)}(\vec{p}) \text{ - outgoing}\end{cases}$

▶ R4 : $\begin{cases} \bar{v}^{(s)} \text{ - incoming} \\ v^{(s)} \text{ - outgoing}\end{cases}$

▶ R5 : The arrows on the fermion lines follow the flow of charge $-e$

▶ R6 : $\begin{cases} \varepsilon_\mu(\vec{k}) \text{ - incoming} \\ \varepsilon_\mu^*(\vec{k}) \text{ - outgoing}\end{cases}$

▶ R7 : $-ie\gamma^\mu$

▶ R8 : include $\delta^4(\sum_i p_i^\mu)$ at each vertex to impose energy-momentum conservation

▶ R9 : compute and assign 4-momenta of all internal propagators, choose direction of momentum flow for photon propagators

!

▶ R10 : For loops :
$\int \frac{dq^4}{(2\pi)^4}$ for unconstrained mom.
(-1) for fermion loops
Tr[---] for closed loops

▶ R11 : $(2\pi)^4\delta^4(P_{out}-P_{in})$ overall

▶ R12 : add all diagrams at given order

▶ R13 : add minus sign to diagrams that differ only in $f_i \leftrightarrow f_i$, $f_f \leftrightarrow f_f$ $f_i \leftrightarrow \bar{f}_f$ or $f_f \leftrightarrow \bar{f}_i$

▶ R14 : the flavour is conserved at the QED vertex

If one includes the longitudinal polarization $\varepsilon_{(3)}^\mu = \frac{1}{m}(|\vec{k}|,0,0,\omega)$, one recovers

$$\boxed{\sum_{\lambda=1}^{③} \varepsilon_\lambda^{\mu,*}\varepsilon_\lambda^\nu = -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}}$$

completeness relation for a virtual photon

Note that $k^\mu J_\mu = k^\mu\bar{u}(p_2)\gamma_\mu u(p_1) \overset{\text{mom. conservation}}{=} \bar{u}(p_2)(p_1^\mu-p_2^\mu)\gamma_\mu u(p_1)$
$= \bar{u}(p_2)(m-m)u(p_1) = 0$ → the term
$-g^{\mu\nu} + \xi\frac{k^\mu k^\nu}{k^2}$ does <u>not</u> yield a contribution!

→ real photon : longitudinal and time-like polarizations cancel

virtual photon : three physical polarization states

→ for practical purposes, the propagator can be reduced to $-i\frac{g^{\mu\nu}}{k^2}$

Massive spin-1 fields exist → $\varepsilon_{(3)}^\mu \rightarrow \frac{E}{M}(1,0,0,1)$

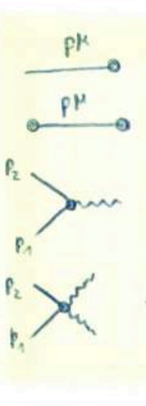
for $E \gg m$ → $\sigma \sim \varepsilon_{(3)}^2 \sim g^2\frac{E^2}{M^2}$ diverges for $E\rightarrow\infty$

↳ the longitudinal degrees of freedom are the result of spontaneous symmetry breaking and are acquired from the Goldstone bosons of the Higgs field

Scalar QED

$$\boxed{\mathcal{L}_{SQED} = (D_\mu\varphi)^\dagger D^\mu\varphi - m^2\varphi^\dagger\varphi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

with $D^\mu = \partial^\mu + ieA^\mu$ → $\mathcal{L}_{int} = ieA_\mu\left[(\partial^\mu\varphi^\dagger)\varphi - \varphi^\dagger\partial_\mu\varphi\right] - e^2A^\mu A_\mu\varphi^\dagger\varphi$



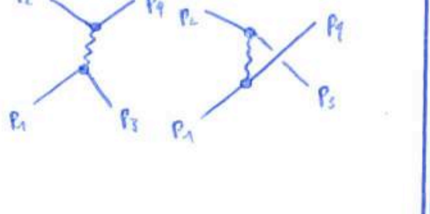
▶ S1 : 1

▶ S2 : $\dfrac{i}{p^2-m_\pi^2+i\varepsilon}$

▶ S3 : $-ie(p_1+p_2)^\mu$

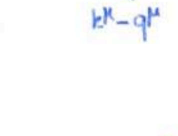
▶ S4 : $-2ie^2g^{\mu\nu}$

$$-i\mathcal{M} = (ie)^2\left[\frac{(p_1+p_3)_\mu(p_2+p_4)^\mu}{(p_1-p_3)^2} + \frac{(p_1+p_4)_\mu(p_2+p_3)^\mu}{(p_1-p_4)^2}\right]$$



Closed loops and self-energies

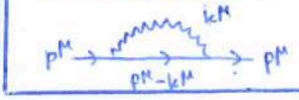
<u>Self-energy</u>:



logarithmic divergence

$$\tilde{G}_{1\text{-loop}}^{\mu\nu}(k^2) = \frac{-ig_{\mu\mu'}}{k^2}\int\frac{dq^4}{(2\pi)^4}\,\mathrm{Tr}\left[(-ie\gamma^{\mu'})\frac{i(\not{q}+m)}{q^2-m^2}(-ie\gamma^{\nu'})\frac{i(\not{k}-\not{q}+m)}{(k-q)^2-m^2}\right]\frac{-ig_{\nu'\nu}}{k^2}\cdot(-1)$$

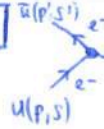
<u>Self-energy of fermions</u>:



The momentum k^μ is unconstrained and we need to perform an integral $\int dk^4$

11. Computations in QED (1/2)

Mott Scattering



Coulomb potential: $\boxed{A^\mu = \left(\frac{Ze}{4\pi|\vec{x}|},0,0,0\right)}$ $\longrightarrow \boxed{S_1} = (-ie)\int dx^4 \langle p',s'|\bar{\Psi}(x)\gamma^0\Psi(x)A_0(x)|p,s\rangle$

$\rightarrow \gamma^\mu A_\mu = \gamma^0 \frac{Ze}{4\pi|\vec{x}|}$

$$= \ldots = \boxed{-i\,2\pi\,\delta(E-E')\,\frac{Ze^2}{q^2}\underbrace{\bar{u}^{(s')}(p')\gamma^0 u^{(s)}(p)}_{\mathcal{M}}}$$

$|\vec{p}-\vec{p}'|$

Solution ①: Plug in the actual spinors $u_\uparrow(p), u_\downarrow(p)$

(see p. 7) $\rightarrow \boxed{\langle|\mathcal{M}|^2\rangle} = \frac{1}{2}\left(|\mathcal{M}_{\uparrow\uparrow}|^2+|\mathcal{M}_{\uparrow\downarrow}|^2+|\mathcal{M}_{\downarrow\uparrow}|^2+|\mathcal{M}_{\downarrow\downarrow}|^2\right)$

$$= \boxed{4\frac{Z^2e^4}{q^4}E^2\left(1-\beta^2\sin^2\left(\frac{\Theta}{2}\right)\right)}$$

$p^\mu = (E,p,0,0)$, $p'^\mu = (E,p\cos\Theta, p\sin\Theta,0)$

(elastic scattering)

Solution ②: Noting that $\left[\bar{u}^{(s)}(\vec{p}')\gamma^\mu u^{(s)}(\vec{p})\right]^\dagger = \bar{u}^{(s)}(\vec{p})\gamma^\mu u^{(s')}(\vec{p}')$

$$\rightarrow \boxed{\langle|\mathcal{M}|^2\rangle} = \frac{1}{2}\frac{Z^2e^4}{q^4}\sum_{s=1}^{2}\sum_{s'=1}^{2}\left[\bar{u}^{(s')}(\vec{p}')\gamma^0 \underbrace{u^{(s)}(\vec{p})\right]\left[\bar{u}^{(s)}(\vec{p})}\gamma^0 u^{(s')}(\vec{p}')\right]$$

trace theorems

$$= \frac{1}{2}\frac{Z^2e^4}{q^4}\operatorname{tr}\left((\not{p}'+m_e\mathbb{1})\gamma^0(\not{p}+m_e\mathbb{1})\gamma^0\right) \quad (\not{p}+m_e\mathbb{1})$$

$$= \boxed{4\frac{Z^2e^4}{q^4}E^2\left(1-\beta^2\sin^2\left(\frac{\Theta}{2}\right)\right)}$$

Cross-section: $\boxed{\left(\frac{d\sigma}{d\Omega}\right) = \frac{(\alpha Z)^2E^2}{4p^4\sin^4\left(\frac{\Theta}{2}\right)}\left(1-\beta^2\sin^2\left(\frac{\Theta}{2}\right)\right)}$ (fixed-target frame)

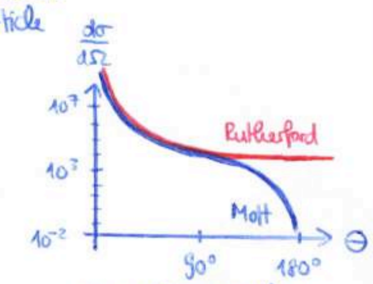
→ scattering off a static potential (no recoil), where the scattered particle is a spin-$\frac{1}{2}$ Dirac particle and relativistic effects are considered

▸ non-relativistic limit: $E \rightarrow m_e$, $E_k = \frac{p^2}{2m_e}$

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right) = \underbrace{\frac{(\alpha Z)^2}{16E_k^2\sin^4\left(\frac{\Theta}{2}\right)}}_{\text{Rutherford}}\underbrace{\left(1-\beta^2\sin^2\left(\frac{\Theta}{2}\right)\right)}_{\text{spin correction}}$$

▸ relativistic limit: $E\rightarrow p$, $\beta \rightarrow 1$

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}\cdot\cos^2\left(\frac{\Theta}{2}\right) \propto \frac{1+\cos\Theta}{(1-\cos\Theta)^2}$$



cross-section vanishes for $\Theta = \pi$, as this would violate angular momentum cons.

Matrix Element $\boxed{-i\mathcal{M}(2\pi)^4\delta(P_{out}-P_{in}) = \text{Feynman expr.}}$

Mandelstam Variables $p+k\rightarrow p'+k'$

$$\boxed{\begin{aligned} s &= (p+k)^2 = (p'+k')^2 \\ t &= (p-p')^2 = (k-k')^2 \\ u &= (p-k')^2 = (k-p')^2 \end{aligned}}$$

$s+t+u = \sum_i m_i^2$

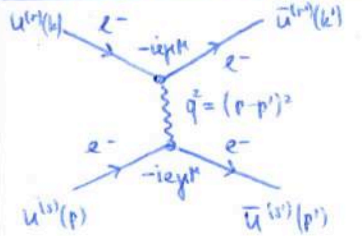
$E_* = \sqrt{s}$

$\rightarrow s \simeq 2pk \simeq 2p'k'$

$t \simeq -2pp' \simeq -2kk'$ (in the ultra-relativistic lim.)

$u \simeq -2pk' \simeq -2kp'$

Electron-lepton scattering $e^-\ell^-\rightarrow e^-\ell^-$

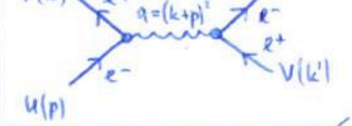


→ Feynman rules → $\langle|\mathcal{M}|^2\rangle$ → Casimir's trick → trace theorems : ultrarel.

$$\boxed{\langle|\mathcal{M}(e^-\ell^-\rightarrow e^-\ell^-)|^2\rangle \simeq 2e^4\left[\frac{s^2+u^2}{t^2}\right]}$$

$$\boxed{\left(\frac{d\sigma(e^-\ell^-\rightarrow e^-\ell^-)}{d\Omega}\right)_{CMS} = \left(\frac{\alpha^2}{2s}\right)\frac{4+(1+\cos\Theta)^2}{(1-\cos\Theta)^2}}$$

Lepton pair creation $e^-e^+\rightarrow \ell^-\ell^+$

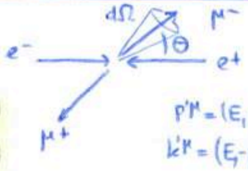


ultrarel.

$$\boxed{\langle|\mathcal{M}(e^-e^+\rightarrow \ell^-\ell^+)|^2\rangle \simeq 2e^4\left[\frac{t^2+u^2}{s^2}\right]}$$

$$\boxed{\left(\frac{d\sigma(e^-e^+\rightarrow \ell^-\ell^+)}{d\Omega}\right)_{CMS} \simeq \left(\frac{\alpha^2}{2s}\right)\frac{(1+\cos^2\Theta)}{2}}$$

Momentum:



$p^\mu = (E,0,0,p)$

$k^\mu = (E,0,0,-p)$

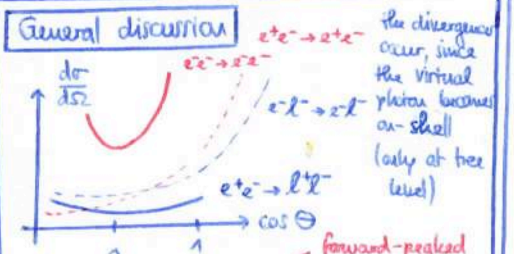
$p'^\mu = (E,p'\sin\Theta,0,p'\cos\Theta)$

$k'^\mu = (E,-p'\sin\Theta,0,-p'\cos\Theta)$

$t \simeq -\frac{s}{2}(1-\cos\Theta)$

$u \simeq -\frac{s}{2}(1+\cos\Theta)$

General discussion



the divergence occur, since the virtual photon becomes on-shell (only at tree level)

forward-peaked

$\frac{1}{t}$ diverges for $p\rightarrow p'$ $(\cos\Theta\rightarrow 1)$

$\frac{1}{u}$ diverges for $p\rightarrow k'$ $(\cos\Theta\rightarrow -1)$

backward-peaked

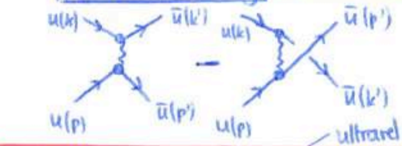
crossing symmetry: $\boxed{k\rightarrow -k',\ k'\rightarrow p',\ p'\rightarrow -k,\ p\rightarrow p}$

$t\rightarrow s$, $u\rightarrow t$, $s\rightarrow u$

crossing symmetry: $\boxed{k\rightarrow -k',\ k'\rightarrow -k}$

$u\rightarrow s$, $s\rightarrow u$, $t\rightarrow t$

Møller scattering $e^-e^-\rightarrow e^-e^-$

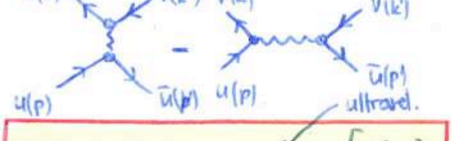


ultrarel.

$$\boxed{\langle|\mathcal{M}(e^-e^-\rightarrow e^-e^-)|^2\rangle \simeq 2e^4\left[\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}+\frac{2s^2}{ut}\right]}$$

$$\boxed{\left(\frac{d\sigma(e^-e^-\rightarrow e^-e^-)}{d\Omega}\right)_{CMS} = \left(\frac{\alpha^2}{2s}\right)\frac{(7+\cos(2\Theta))^2}{2\sin^4\Theta}}$$

Bhabha scattering $e^-e^+\rightarrow e^-e^+$



ultrarel.

$$\boxed{\langle|\mathcal{M}(e^-e^+\rightarrow e^-e^+)|^2\rangle \simeq 2e^4\left[\frac{s^2+u^2}{t^2}+\frac{u^2+t^2}{s^2}+\frac{2u^2}{ts}\right]}$$

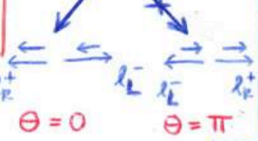
$$\boxed{\left(\frac{d\sigma(e^+e^-\rightarrow e^+e^-)}{d\Omega}\right)_{CMS} = \left(\frac{\alpha^2}{2s}\right)\frac{(3+\cos^2\Theta)^2}{2(\cos\Theta-1)^2}}$$

Helicity Conservation

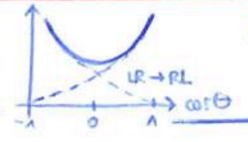
Consider $e^-e^+\rightarrow e^-e^+$ for $E \gg m$

One can calculate individual helicity contributions individually using Casimir's trick and $\bar{v}(k)\gamma^\mu\left(\frac{1\pm\gamma^5}{2}\right)u(p)$

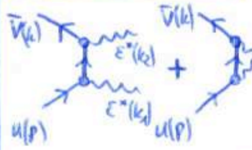
$$\boxed{\begin{aligned} \frac{d\sigma}{d\Omega}(e_L^+e_R^-\rightarrow \ell_L^+\ell_R^-) &= \frac{d\sigma}{d\Omega}(e_R^+e_L^-\rightarrow \ell_R^+\ell_L^-) = \frac{\alpha^2}{4s}(1+\cos\Theta)^2 \\ \frac{d\sigma}{d\Omega}(e_R^+e_L^-\rightarrow \ell_L^+\ell_R^-) &= \frac{d\sigma}{d\Omega}(e_L^+e_R^-\rightarrow \ell_R^+\ell_L^-) = \frac{\alpha^2}{4s}(1-\cos\Theta)^2 \end{aligned}}$$



→ in total : $\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\Theta)}$

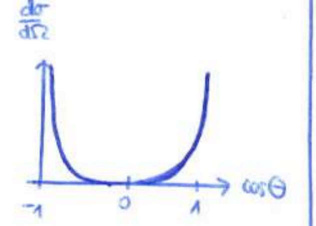


Pair annihilation $e^-e^+ \longrightarrow \gamma + \gamma$



Ward identity: $k_{1\nu}M^{\mu\nu} = \underbrace{k_{1\nu}M_1^{\mu\nu}}_{\neq 0} + \underbrace{k_{1\nu}M_2^{\mu\nu}}_{\neq 0} = 0$

$$k_{2\mu}M^{\mu\nu} = \overbrace{k_{2\mu}M_1^{\mu\nu}} + \overbrace{k_{2\mu}M_2^{\mu\nu}} = 0$$



Using Casimir's trick and $\sum_{\lambda=1}^{2} \varepsilon_\lambda^\mu \varepsilon_\lambda^{*\nu} = -g^{\mu\nu}$

ultrarel.

$$\longrightarrow \boxed{\langle |M|^2 \rangle \simeq 2e^4 \left[\frac{u}{t} + \frac{t}{u}\right]}$$

ultrarel.

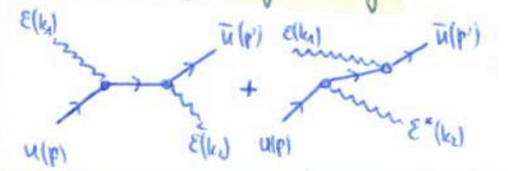
$$\boxed{\frac{d\sigma(e^+e^- \to \gamma\gamma)_{ur}}{d\Omega} \simeq \frac{\alpha^2}{s}\left(\frac{1+\cos^2\Theta}{\sin^2\Theta}\right)}$$

$$\boxed{\frac{d\sigma(e^+e^- \to \gamma\gamma)_{nr}}{d\Omega} \simeq \frac{\alpha^2}{4m_e p}}$$

incoming momentum

→ isotropic distribution

Compton Scattering $e^-\gamma \longrightarrow e^-\gamma$



Klein – Nishina: $$\boxed{\left(\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega}\right)_{lab,\,unpol.} = \frac{r_e^2}{2}\left(\frac{k'}{k}\right)^2\left[\frac{k'}{k} + \frac{k}{k'} - \sin^2\Theta\right]}$$

electron radius $r_e = \frac{\alpha}{m_e}$

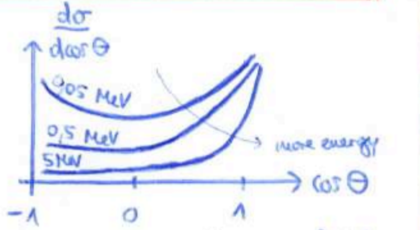
momentum of outgoing γ

momentum of incoming γ

Compton: $$\boxed{\left(\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\cos\Theta}\right)_{lab,\,unpol.} = \pi r_e^2 \frac{1}{[1+\alpha(1-\cos\Theta)]^2}\left[1 + \cos^2\Theta + \frac{\alpha^2(1-\cos\Theta)^2}{1+\alpha(1-\cos\Theta)}\right]}$$

$\frac{k}{m_e}$

↳ describes the scattering off quasi-free atomic electrons → incoherent scattering

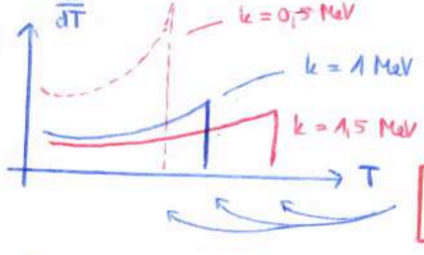


11. Computations in QED (2/2)

$\frac{\alpha}{m_e}$ $\frac{k}{m_e}$

T ← kin. energy of electron
k ← incoming photon energy

$$\boxed{\left(\frac{d\sigma}{dT}\right) = \frac{\pi r_e^2}{m_e \alpha^2}\left[2 + \frac{x^2}{\alpha^2(1-x)^2} + \frac{x}{1-x}\left(x - \frac{2}{\alpha}\right)\right]}$$



Compton edge: the photon is back-scattered

$$\boxed{T_{max} = k\,\frac{2\alpha}{1+2\alpha}}$$

$$\sigma_{tot}(k) = Z\int_{T_{min}}^{T_{max}} dT\left(\frac{d\sigma}{dT}\right) = 2\pi Z r_e^2\left[\frac{(\alpha^2-2\alpha-2)}{2\alpha^3}\ln(1+2\alpha) + \frac{\alpha^3+9\alpha^2+8\alpha+2}{4\alpha^4+4\alpha^3+\alpha^2}\right]$$

- high energies: $k \to \infty$, $\alpha \gg 1$: $\boxed{\sigma_{tot} \to 0}$
- low energies: $k \to 0$, $\alpha \to 0$: $\boxed{\sigma_{tot} = \frac{8}{3}\pi Z r_e^2}$

↳ Thomson scattering: incoherent scattering of photons by free electrons in the classical limit

↳ for $k \lesssim 100\,eV$, the binding energy of the atomic electrons must be taken into account:

$$\left(\frac{d\sigma}{dT}\right) = \left(\frac{d\sigma}{dT}\right) \cdot S(k,k')$$ ← scattering function

Rayleigh scattering: coherent scattering of photons by the whole atom

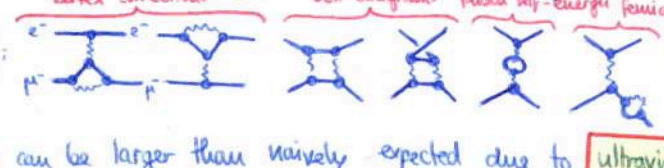
← electrons participate in coherent manner

12. QED Radiative Corrections (1/2)

Loop Contributions We've seen that QFT predictions can be found perturbatively: $S = 1 + S_1 + S_2 + \dots$ with $S_i \sim e^{2i} \sim \alpha^i$ ($\rightarrow$ since $\alpha \simeq \frac{1}{137}$, we can do perturbation theory). The error of calculation can be determined via $|P(\alpha^{(n-1)}) - P(\alpha^{(n)})|$.

In addition to the tree-level diagrams, we always get higher-order corrections:

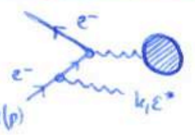
$e^-\mu^- \rightarrow e^-\mu^-$:



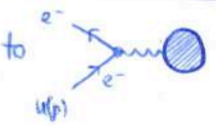
$e^-\mu^- \rightarrow e^-\mu^-\gamma$:

These corrections can be larger than naively expected due to $\boxed{\text{ultraviolet, infrared or collinear singularities.}}$

Soft Photons



correction to



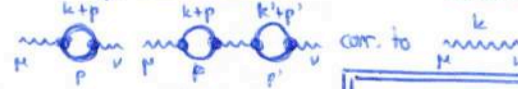
$$M(p,k) = iM_0(p-k)\frac{\not p - \not k + m_e}{(p-k)^2 - m_e^2}(-ie\gamma^\mu)u(p,s)\varepsilon_\mu^*(k)$$

$$\simeq -e\left(\frac{p\cdot\varepsilon^*}{p\cdot k}\right)M_0(p)u(p,s) \quad \text{for } |\vec k| \ll |\vec p| \; \left(\begin{smallmatrix}\text{soft-photon}\\ \text{limit}\end{smallmatrix}\right)$$

$\rightarrow$ the correction becomes large for $p\cdot k \rightarrow 0$

- ▶ $|\vec k| \rightarrow 0$: $\boxed{\text{infrared divergence}}$ $\rightarrow$ momentum threshold (sum over soft photon contrib. and add an overall soft-photon correction)
- ▶ $\vec k \parallel \vec p$ and $E_p \gg m_e$: $\boxed{\text{collinear divergence}}$ $\rightarrow$ angle threshold

Self-energy of the photon



photon propagator correction

$$G^{\mu\nu}_{F,1\text{-}L} = \left(\frac{-ig^{\mu\nu}}{k^2}\right)\underbrace{(-1)\int\frac{dp^4}{(2\pi)^4}\mathrm{Tr}\left[(ie\gamma_\alpha)\left(\frac{-i(\not k + \not p + m)}{(k+p)^2 - m^2}\right)(ie\gamma_\beta)\left(\frac{-i(\not p + m)}{p^2 - m^2}\right)\right]}_{\boxed{ie^2\Pi^{1L}_{\alpha\beta}(k)}}\left(\frac{-ig^{\alpha\beta}}{k^2}\right)$$

$$G^{\mu\nu}_{F,2\text{-}L} = (-e^2)\tilde G_F^{\mu\alpha}(k)\,\Pi^{1L}_{\alpha\beta}(k)\,\tilde G_F^{\beta\gamma}(k)\,\Pi^{1L}_{\gamma\delta}(k)\,\tilde G_F^{\delta\nu}(k)$$

$$\Pi^{1L}_{\alpha\beta}(k) \sim \int dp^4\frac{p^2}{p^4} \sim \int dp\cdot p \sim p^2 \longrightarrow \infty \quad \boxed{\text{ultraviolet divergence}}$$

$\rightarrow$ in order to calculate $\Pi^{\alpha\beta}_{1L}$, we need to introduce some regularization:

$$\Pi^{\alpha\beta}_{1L}(k,\Lambda) = i\int\frac{dp^4}{(2\pi)^4}\mathrm{Tr}[\dots]\boxed{\frac{\Lambda^4}{(\Lambda^2 - p^2)^2}}$$

cutoff: we basically exclude the contrib. at very high energy, as these are anyways unknown

$\rightarrow$ generally: $\Pi^{\alpha\beta}_{1L}(k,\Lambda) = g^{\alpha\beta}B_{00}(k^2,\Lambda) + k^\alpha k^\beta B_{11}(k^2,\Lambda)$

does not contribute to a typical vector current

with $B_{00}(k^2,\Lambda) = \underbrace{A_0(\Lambda)}_{\rightarrow \Lambda^2} + \underbrace{k^2A_1(\Lambda)}_{\rightarrow \log\Lambda} + \underbrace{k^4A_2(k^2,\Lambda)}_{\rightarrow \text{finite}}$

Here $$\boxed{ie^2\Pi^{\alpha\beta}(k,\Lambda) = -ig^{\alpha\beta}k^2\left[\left(\frac{\alpha}{3\pi}\right)\left(\log\left(\frac{\Lambda^2}{m^2}\right) - f\left(-\frac{k^2}{m^2}\right)\right)\right]}$$

with $f(x) = 6\int_0^1 dz\, z(1-z)\log(1 + xz(1-z)) \longrightarrow \begin{cases}\log x, & x \gg 1\\ \frac{x}{5}, & x \ll 1\end{cases}$

$$\rightarrow \boxed{\tilde G^{\mu\nu}(k,\Lambda) = \left(\frac{-ig^{\mu\nu}}{k^2}\right)\left[1 - \frac{\alpha}{3\pi}\left(\log\left(\frac{\Lambda^2}{m^2}\right) - f\left(-\frac{k^2}{m^2}\right)\right)\right]}$$

Renormalization

In order to get finite predictions from the theory, one has to deal with the infinities that occur in a consistent way $\rightarrow$ $\boxed{\text{renormalization}}$

The electroweak theory and QCD are renormalizable, there are two schemes: the $\boxed{\text{minimal subtraction scheme}}$ and the $\boxed{\text{on-shell renormalization scheme}}$

Renormalization gives finite observables at the energy scales we are interested in without knowing the exact details of the theory at very high energies.

Electric charge renormalization: e_0, m_0 are the bare parameters that show up in the Lagrangians, e, m are the values we can measure.

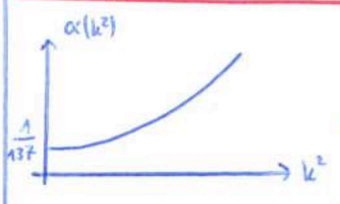
We tune e_0 such that we find the right e:

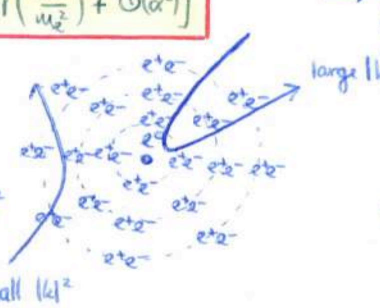
$$\boxed{e = e_0\sqrt{1 - \frac{e_0^2}{12\pi^2}\ln\left(\frac{\Lambda^2}{m_e^2}\right)}}$$

The dependence on k remains:

$$\boxed{e(k^2) = e(0)\sqrt{1 + \frac{e^2(0)}{12\pi^2}f\left(-\frac{k^2}{m_e^2}\right) + \mathcal{O}(e^4)}}$$

$$\boxed{\alpha(k^2) = \alpha(0)\left[1 + \frac{\alpha(0)}{3\pi}f\left(\frac{-k^2}{m_e^2}\right) + \mathcal{O}(\alpha^2)\right]}$$



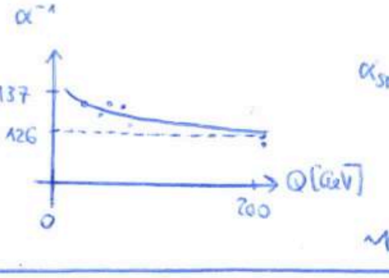


Experiment: To get real values, α is measured at a given $\alpha(\mu^2)$

$$\rightarrow \boxed{\alpha(Q^2) = \alpha(\mu^2)\left[1 + \frac{\alpha(\mu^2)}{3\pi}\cdot\log\left(\frac{Q^2}{\mu^2}\right) + \dots\right] = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{2\pi}\log\left(\frac{Q^2}{\mu^2}\right)}}$$

$\rightarrow$ tests are interesting to see whether the concepts of renormalization hold

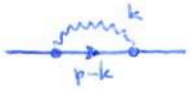
$$\alpha(Q^2 \simeq 0) = \frac{1}{137{,}03\dots \pm \underbrace{0{,}000000032}_{10^{-8}}}$$ (measured in e^-e^+ collisions) ↳ DORIS, PEP, TRISTAN, LEP



$$\alpha_{SM}(s) = \frac{\alpha(0)}{1 - \Delta\alpha_{lepton} - \Delta\alpha_{hadr.} - \Delta\alpha_{top} - \Delta\alpha_{new}}$$

- $\Delta\alpha_{lepton}$: lepton loops, $\sim 10^{-2}$
- $\Delta\alpha_{hadr.}$: quark loops, $\sim 10^{-2}$ ↳ can be related to cross-section for hadron production in e^-e^+ annihilation
- $\Delta\alpha_{top}$: top quark loops
- $\Delta\alpha_{new}$: no evid. found

Self-energy of the electron



correction to

$$\tilde{S}_{F,1\text{-}L}(p) = \frac{i(\not{p}+m)}{p^2-m^2}\underbrace{\int\frac{dk^4}{(2\pi)^4}\left[(-ie\gamma^\alpha)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}(-ie\gamma^\beta)\frac{-ig_{\alpha\beta}}{k^2}\right]}_{\boxed{ie^2\Sigma^{1L}(p)}}\frac{i(\not{p}+m)}{p^2-m^2}$$

fermion propagator correction

$$\Sigma^{1L}(p) \sim \int dk^4\,\frac{k}{k^4} \sim \int dk \sim k \to \infty \quad \boxed{\text{ultraviolet divergence}}$$

→ we again introduce a cutoff to be able to calculate the propagator:

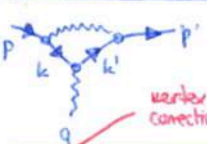
$$\Sigma_{1L}(p,\Lambda) = i\int\frac{dk^4}{(2\pi)^4}\cdots\boxed{\left[\frac{\Lambda^2}{k^2+\Lambda^2}\right]}$$

→ generally : $\Sigma_{1L}(p,\Lambda) = A_0(\Lambda) + \underbrace{(\not{p}-m)A_1(\Lambda) + (\not{p}-m)^2A_2(p,\Lambda)}_{\text{do not contribute to } \bar{u}(p)\Sigma_{1L}u(p)}$

→ We get a term $\bar{u}(p)A_0(\Lambda)u(p)$, which is equivalent to adding $A_0(\Lambda)$ to the mass : $\boxed{m = m_0 + \bar{u}(p)A_0(\Lambda)u(p)}$

Here : $\boxed{\delta m} = ie_0^2\,\Sigma_{1L}(\Lambda) = \boxed{\left(\frac{3\alpha}{2\pi}\right)m_0\log\left(\frac{\Lambda}{m_0}\right)}$

QED vertex corrections



correction to

vertex correction $\bar{u}(p')\Gamma u(p)$

$$\Gamma^\mu_{1L} = \int\frac{dk^4}{(2\pi)^4}\frac{-ig_{\nu s}}{(k-p)^2}(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{(k'^2-m^2)}\gamma^\mu\frac{i(\not{k}+m)}{(k^2-m^2)}(-ie\gamma^s)$$

→ generally : $(ie)\bar{u}(p')\Gamma^\mu u(p) = \bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2)\right]u(p)$ with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$

structure factor

structure factor

Here : $\boxed{F_1(q^2) = 1 + \mathcal{O}(\alpha,q^2)}$ $\boxed{F_2(q^2) = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha,q^2)}$

→ $(ie)\bar{u}(p')\Gamma^\mu u(p) = (ie)\bar{u}(p')\left[\frac{1}{2m}(p'+p)^\mu + \frac{i}{2m}\left(1+\frac{\alpha}{2\pi}\right)\sigma^{\mu\nu}q_\nu + \ldots\right]u(p)$

12. Radiative Corrections (2/2)

13. Tests of QED at High Energy

QED Breakdown Assumption: QED embedded in a more general theory characterized by an as yet inaccessible energy region → fictitious "cutoff" Λ up to which the theory is holding

Excited electron model: Electron has inner structure → vertex $e^*e\gamma$

$$\boxed{\mathcal{L}_{int,e^*} = \frac{e}{2\Lambda}\overline{\Psi}_{e^*}\sigma^{\mu\nu}\Psi_e F_{\mu\nu} + h.c.}$$

$$\uparrow \boxed{\Lambda = \frac{m_{e^*}}{\lambda}}$$

General class of BSM models:

$$\boxed{\frac{d\sigma}{d\Omega}(e^+e^- \to \gamma\gamma) = \left(\frac{d\sigma}{d\Omega}\right)_{QED}\left[1 \pm \frac{s^2}{2\Lambda_\pm^4}\sin^2\Theta\right]}$$

Lepton Colliders e^+e^- accelerators: $\underbrace{\text{SPEAR, PEP, SLC}}_{\text{SLAC}}$, $\underbrace{\text{DORIS, PETRA}}_{\text{DESY}}$, $\underbrace{\text{TRISTAN}}_{\text{KEK}}$, $\underbrace{\text{LEP}}_{\text{CERN}}$

Advantages: collider setup → high $\sqrt{s}$
e^+e^- → clean collisions

Interactions: $e^+e^- \to e^+e^-$, $e^+e^- \to \mu^+\mu^- / \tau^+\tau^-$, $e^+e^- \to \gamma\gamma$, $e^+e^- \to q\overline{q} \to$ hadrons
$e^+e^- \to e^+e^- f\overline{f}, \dots$

Detectors:
① **Tracking**: measure points along trajectories of charged particles (→ number, point of origin, direction of travel, momentum)
② **Electromagnetic Calorimetry**: photon / electron → electromagnetic cascade → total energy can be inferred
alternating layers of high Z / low Z materials
③ **Hadronic Calorimetry**: estimate total energy of all particles, more material used → even hadrons will cascade into lower energy particles
④ **Muon Detectors**: placed around all previous elements → only reached by very penetrating particles like muons
⑤ **Particle Identification**: to identify charged and long-lived particles: measure p and v
→ Cherenkov counters, $\frac{dE}{dx}$, time of flight over known distance, ...

Testing QED and Electroweak effects

QED embedded in electroweak theory
→ contributions become relevant for $s, q^2 \sim m_{Z^0}^2$

For $e^+e^- \to \ell^+\ell^-$
$e^+e^- \to e^+e^-$
$e^+e^- \to \gamma\gamma$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{meas.} = \left(\frac{d\sigma}{d\Omega}\right)_{QED,\,Born}\left(1 + \delta_{radcorr.} + \delta_{electroweak}\right)}$$

exp. agree very well with QED
→ no effect of weak theory established

at the ~1% level before LEP, SLC

Total cross-section $e^+e^- \longrightarrow \mu^+\mu^-$

$$\boxed{\sigma_{tot}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}$$

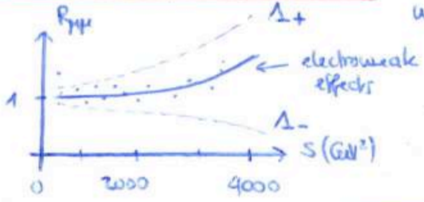
from phase-space factor and thus generally true for processes involving point-like initial state particles
→ luminosities of colliders must increase with energy

→ it is $\frac{86{,}8\text{ nb}}{E_x^2\,(\text{GeV}^2)}$

if there were new physics

$$\boxed{R_{\mu\mu}(s) := \frac{\sigma_{\mu\mu}(s)}{\sigma_{\mu\mu}^{QED}(s)} = \left(1 \mp \frac{s}{s - \Lambda_\pm^2}\right)^2}$$

lower bounds for $\Lambda_\pm$ of 250 GeV were found



no significant deviations from QED were found

Photon pair production $e^+e^- \longrightarrow \gamma\gamma$

- provides a **clean test of QED**, as the contributions from the electroweak theory are small
- at higher energies, Bremsstrahlung becomes relevant, hence studies have been extended to $e^+e^- \to \gamma\gamma(n\gamma)$

$$\Lambda_+ > 321\text{ GeV}, \quad \Lambda_- > 282\text{ GeV}, \quad m_{e^*} > 283\text{ GeV}$$

(at 95% CL)

Bhabha Scattering $e^+e^- \to e^+e^-$ shows large t-channel contributions for small angles
→ **determine the luminosity** in e^+e^- experiments

Background: $e^+e^- \to$ hadrons: not all energy is deposited in electromagnetic calorimeter

$e^+e^- \to \begin{matrix} e^+e^-e^+e^- \\ \tau^-\tau^+ \to e^+e^- + \nu_\tau \end{matrix}$: no colinearity, momentum missing

$\boxed{e^+e^- \to \gamma\gamma}$: tough, since γ can change to e^+e^- → needs to be handled on a statistical basis

QED predictions agree very well with experiments → composite structure of e^- can be excluded to 10^{-18} m

Forward-Backward Asymmetry $e^+e^- \longrightarrow \mu^+\mu^-$

$$\boxed{\frac{d\sigma}{d\cos\Theta} = \frac{\pi\alpha^2}{2s}\left[R_{\mu\mu}(1 + \cos^2\Theta) + B\cos\Theta\right]}$$

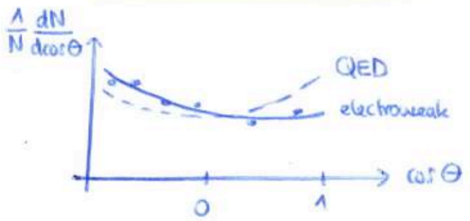
Theory:

$$A_{FB} = \frac{\int_{\cos\Theta > 0} d\Omega\,\frac{d\sigma}{d\Omega} - \int_{\cos\Theta < 0} d\Omega\,\frac{d\sigma}{d\Omega}}{\int d\Omega\,\frac{d\sigma}{d\Omega}}$$

→ easy to compute / measure

Experiment:

$$A = \frac{N(\Theta < 90°) - N(\Theta > 90°)}{N(\Theta < 90°) + N(\Theta > 90°)}$$



provided strong support for the predictions of the electroweak theory

Magnetic Moment of Dirac Particle Using minimal substitution for the Dirac Hamiltonian:

$$i\partial_t\psi = \left(\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta m + e\phi\right)\psi \rightarrow \text{Ansatz } \psi(x) = e^{-imx^0}\begin{pmatrix}\psi_A(x)\\ \psi_B(x)\end{pmatrix} \rightarrow \underbrace{i\partial_0\psi_A \simeq \left(\frac{(\vec{p}-e\vec{A})^2}{2m}-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right)\psi_A(x)}_{\text{low-energy limit}}$$

$$-\frac{e}{m}\vec{\sigma}\cdot\vec{B} = \boxed{-\underbrace{2}_{g_{Dirac}}\underbrace{\frac{e}{2m}}_{\mu_B \text{ Bohr magneton}}\underbrace{\frac{1}{2}\vec{\sigma}}_{\vec{S}}\cdot\vec{B}} \longrightarrow \boxed{g_{Dirac} = 2}$$

true for a "bare" point-like Dirac particle

g-factor, Landé factor or dimensionless magnetic moment

$$\boxed{-\vec{\mu}_{Dirac} = g_{Dirac}\mu_B\vec{S}}$$

14. Tests of QED at Low Energy (1/2)

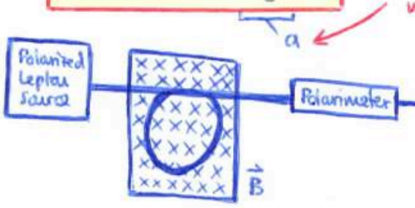
Electron Magnetic Moment

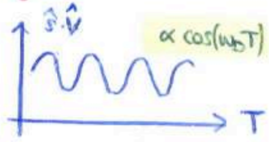
Precession Experiments: observation of spin precession of polarized electrons (muons) in a constant magnetic field

Cyclotron motion: $\boxed{\omega_c = \frac{1}{\gamma}\cdot\underbrace{\frac{eB}{m}}_{\omega_0}}$ Spin motion (Larmor precession): $\boxed{\omega_s = \underbrace{\frac{g}{2}\gamma\omega_c}_{\text{precession at rest}} + \underbrace{(1-\gamma)\omega_c}_{\text{Thomas precession due to acc. of circular motion}}}$

$$\rightarrow \boxed{\omega_D \equiv \omega_s - \omega_c = \underbrace{\frac{g-2}{2}}_{a}\omega_0}$$

anomalous magnetic moment





Spin precesses with ω_s, velocity precesses with ω_c

→ increase T for more precision

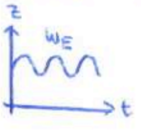
→ in practice, one also needs an electric field to keep electrons on their tracks → BMT equation

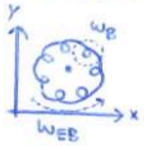
Resonance Experiments: confinement of low-energy (0,01-10 eV) electrons in a **Penning trap**

axial magnetic field: $\vec{B} = B_0\hat{z}$ → radial confinement

electric quadrupole field: $V(r,z) = \left(\frac{V_0}{r_0^2}\right)\left(\frac{r^2}{2}-z^2\right)$ → axial confinement

Three decoupled harmonic oscillations:





① cyclotron $\omega_B = \omega_0 - \omega_{EB}$ ~12 GHz

② axial ω_E ~40 MHz

③ drift of cyclotron orbit center ω_{EB} ~70 kHz

$$\omega_0 = \frac{eB}{m} \quad \omega_E = \sqrt{\frac{2eV_0}{mr_0^2}}$$

$$\omega_{EB} = \frac{\omega_0}{2} - \sqrt{\frac{\omega_0^2}{4}-\frac{\omega_E^2}{2}} \simeq \frac{\omega_E^2}{2\omega_0}$$

$$\rightarrow \boxed{E(n_B,n_E,n_{EB},S_z) = \left(n_B+\tfrac{1}{2}\right)\omega_B + \left(n_E+\tfrac{1}{2}\right)\omega_E - \left(n_{EB}+\tfrac{1}{2}\right)\omega_{EB} - \overbrace{(1+a)\omega_0}^{\omega_s}S_z}$$

→ to measure a, measure $\omega_s - \omega_B$ $(\Delta n_B = \pm 1, \Delta S_z = \mp 1)$ and either ω_r or ω_0 → to know $\vec{B}$

Value: The measurements of a_e were always in good agreement with the theoretical predictions.

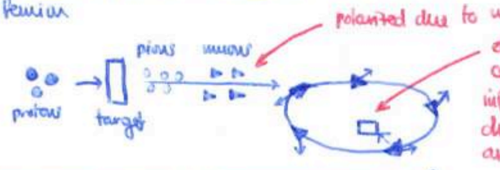
$$a_e(\text{theo.}) = A_e\left(\frac{\alpha}{\pi}\right)^2 + \ldots + E_e\left(\frac{\alpha}{\pi}\right)^{⑤} + \ldots$$

Dyson expansion up to fifth order!

$$= 0{,}001159652181643(25)_{De}(23)_{Ee}(16)_{\text{hadr + EW}}(763)_{\alpha}$$

$a_e(\exp) = 0{,}00115965218091(26)$ $(2\cdot10^{-10}$ relative precision$)$

Muon Magnetic Moment Difference between a_e, a_μ stems from the dependence of the vacuum polarization terms on the mass of the fermion



polarized due to momentum selection

electrons emitted above a certain energy contains information about the spin direction → more electrons are emitted in spin direction

$$\boxed{\vec{\omega}_a = \vec{\omega}_s - \vec{\omega}_c = \frac{e}{m_\mu}\left[a_\mu\vec{B} - a_\mu\frac{\gamma}{\gamma+1}\overbrace{(\vec{\beta}\cdot\vec{B})}^{0}\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)(\vec{\beta}\times\vec{E})\right]}$$

→ expression for $\vec{\omega}_a$ more complicated, since there is a superposition of an electric and a magnetic field (→ BMT formula)

$\vec{\omega}_a\big|_{magic} = a_\mu\frac{e\vec{B}}{m_\mu}$, where the magic momentum is defined via $\boxed{a_\mu - \frac{1}{\gamma^2-1} = 0}$

$a_\mu(\text{theo.}) = 0{,}0011659181(8)$ $(7\cdot10^{-7}$ relative precision$)$

↳ QED, hadronic, weak contribution

↳ either from $e^+e^- \rightarrow$ hadrons (dispersion integral) or use data from $\tau \rightarrow$ hadrons decays together with conserved vector current hypothesis and appropriate isospin corrections

the two approaches do not agree!

$a_\mu(\exp) = 0{,}0011659208(6)$ $(5\cdot10^{-7}$ relative precision$)$

→ $\boxed{2{,}7\sigma/1{,}4\sigma \text{ discrepancy}}$ between experiment and theory!

Bound States – Hydrogenic Atoms

Hydrogenic atoms: hydrogen (e^-p^+), positronium (e^-e^+), muonium $(e^-\mu^+)$, pionic hydrogen (π^-p^+), muonic hydrogen (μ^-p^+)

Atomic states: $\boxed{n^{2s+1}X_j}$ ← S, P, D, F, G, ...

Dirac fine structure $H_{DC} = \vec{\alpha}\vec{p} + \beta m\mathbb{1} + V(r)\mathbb{1}$ with $V(r) = -\frac{Ze^2}{4\pi r}$

→ introducing spherical harmonics spinors: $E_{nj} = m\left[1 + \frac{Z^2\alpha^2}{\left(n-(j+\frac{1}{2})+\sqrt{(j+\frac{1}{2})^2-Z^2\alpha^2}\right)^2}\right]^{-\frac{1}{2}}$

→ Expanding in powers of α: $\boxed{E_{nj} \simeq m\left[\underbrace{1}_{\text{rest energy}} \underbrace{- \frac{Z^2\alpha^2}{2n^2}}_{\text{Bohr levels}} \underbrace{- \frac{(Z^2\alpha^2)^2}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)}_{\text{fine structure splitting}} + \ldots\right]}$

does not include the Lamb shift

Atomic structure of Hydrogen:

① **Bohr levels** (n): – classical QM – Coulomb potential

② **Dirac fine structure** (n, j): – rel. correction – spin-orbit coupling – Darwin term

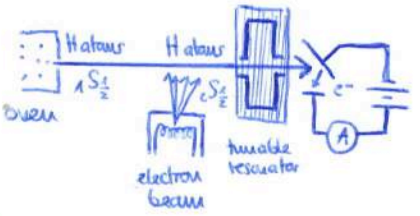
③ **Lamb shift**: – $g_e \neq 2$ – screening effects → $E_{2S_{\frac{1}{2}}} \neq E_{2P_{\frac{1}{2}}}$

④ **Hyperfine splitting**: – nuclear spin interacting with orbital and electron spin

The Lamb shift QED is needed to explain the Lamb shift.

1938 : ${}^3P_{\frac{1}{2}} \rightarrow {}^2S_{\frac{1}{2}}$ showed larger separation than predicted by Dirac theory

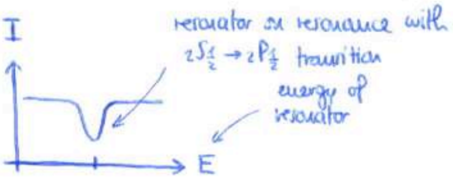
1947 : energy difference between ${}^2S_{\frac{1}{2}}, {}^2P_{\frac{1}{2}}$ measured by Lamb and Retherford



The resonator can induce $2S_{\frac{1}{2}} \rightarrow 2P_{\frac{1}{2}}$

→ if the atoms are in the $2P_{\frac{1}{2}}$ state

here, they decay to the ground state before hitting the foil, otherwise they liberate electrons from the foil by Auger emission. $2S_{\frac{1}{2}} \rightarrow 1S_{\frac{1}{2}}$ prohibited



Positronium $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_e}{2}$ for positronium, hence $m_e \rightarrow \frac{m_e}{2}$, $E_n \rightarrow \frac{1}{2} E_n$, $a_0 \rightarrow 2a_0$

since positronium is free of finite-size effects, it has proven to be an ideal and clean system for testing the accuracy of bound state QED calculations

Ortho- vs. para-positronium : $\boxed{\text{Ortho : } (\uparrow\uparrow), \text{ para : } (\uparrow\downarrow)}$,

$$\left.\begin{aligned} \Delta E_{p\text{-}Ps} &= -\frac{1}{64} m\alpha^4 - \frac{1}{4} m\alpha^4 \\ \Delta E_{o\text{-}Ps} &= \underbrace{-\frac{1}{64} m\alpha^4}_{\text{relativistic correction}} + \underbrace{\frac{1}{4} m\alpha^4}_{\text{spin-spin coupling}} + \underbrace{\frac{1}{12} m\alpha^4}_{\text{emission and re-absorption of a virtual photon}} \end{aligned}\right\} \Delta E_{HFS} = -\frac{7}{12} m\alpha^4$$

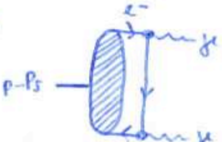
Lifetime calculations : $\left.\begin{aligned} P &= (-1)^{\ell+1} \\ C &= (-1)^{\ell+s} \end{aligned}\right\} CP = (-1)^{s+1}$ $\qquad C(n\gamma) = (-1)^n$

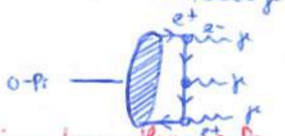
$$\rightarrow \boxed{\begin{aligned} p\text{-}Ps &\rightarrow \gamma\gamma \\ o\text{-}Ps &\rightarrow \gamma\gamma\gamma \end{aligned}}$$

Wheeler-Pirenne : $\Gamma(Ps \rightarrow n\gamma) = \frac{1}{2J+1} \overbrace{|\phi(0)|^2}^{\text{bound state physics}} \underbrace{\left(4v_{rel}\, \sigma(e^+e^- \rightarrow n\gamma)\right)}_{\text{scattering process}}$

$$\rightarrow \boxed{\begin{aligned} \Gamma^{th}(p\text{-}Ps \rightarrow \gamma\gamma) &= \frac{\alpha^5 m_e}{2} \simeq 8032{,}5\ \mu s^{-1} \\ \Gamma^{th}(o\text{-}Ps \rightarrow \gamma\gamma\gamma) &= \frac{2}{9}(\pi^2 - 9)\frac{\alpha^6 m_e}{\pi} \simeq 7{,}2111\ \mu s^{-1} \end{aligned}}$$

↳ o-Ps lives ~1000 times longer than p-Ps, since the latter has an additional factor of α in Γ and a reduced phase space





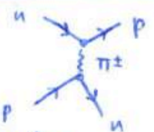
14. Tests of QED at Low Energy (2/2)

o-Ps lifetime puzzle: Theoretical predictions for the lifetime of o-Ps used to be different from the measurements, but now there is an excellent agreement : $\Gamma^{exp}(o\text{-}Ps) = 7{,}0404(10)^{stat}(8)^{syst.}\ \mu s^{-1}$

15. Hadrons (1/2)

Meson

What holds nuclei together? → strong force postulated



Yukawa: meson as gauge boson for strong force with screened Coulomb potential $\boxed{U(r) \propto g^2 \frac{e^{-\lambda r}}{r}}$

$\lambda \simeq \frac{1}{2\hbar m}$ for strong force → $c\Delta t \simeq \lambda^{-1}$ and

$\Delta E \Delta t \simeq \hbar \rightarrow m_{\pi^\pm} \simeq 100\,\text{MeV} \rightarrow m_e < m_{\pi^\pm} < M_p, M_n$

→ hence the name meson

Cosmic Rays

Energy spectrum follows a power law: $\Phi_p(E) \propto E^{-\gamma_p}$ ← 2,7

Hadronic shower cascade: $NN \longrightarrow NN + n\pi s$ (N ← nucleon)

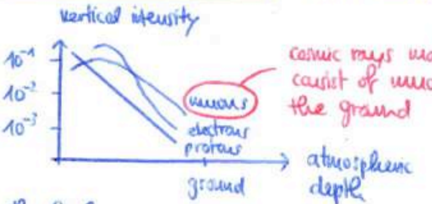
$\hookrightarrow \pi^0 \rightarrow \gamma + \gamma \rightarrow e^-, e^+, \dots$

$\pi^+ \rightarrow \mu^+ + \nu_\mu$

$\hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu$

$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$

$\hookrightarrow e^- + \bar{\nu}_e + \nu_\mu$

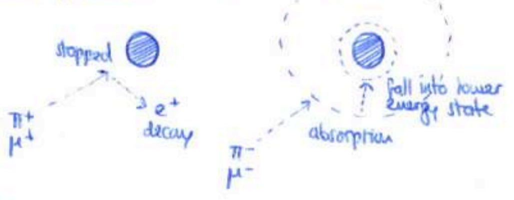


cosmic rays mostly consist of muons on the ground

the higher a muon energy, the higher the probability that it reaches the surface before decaying

Pion or Muon Capture

Positive particles are repelled by nuclei, but negative particles can be captured



interactions between the nucleus and the captured particle take place

$$r(Z) = \left(\frac{m_e}{m_{\text{particle}}}\right) \frac{a_0 n^2}{Z}$$

very close to nucleus

- pions: strong interactions
- muons: $\mu^- + p \rightarrow n + \nu_\mu\ (+\gamma)$

$\boxed{\tau^{-1} = \tau_{\text{decay}}^{-1} + \tau_{\text{capture}}^{-1}}$ → negative particles have a shorter lifetime

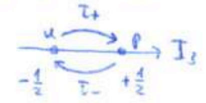
Isospin Symmetry

$M_n - M_p = 1{,}3\,\text{MeV}$ (hence $n \rightarrow p + e^- + \bar{\nu}_e$ possible)

$\boxed{I = \frac{1}{2} \quad p = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \quad n = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle}$

analogous to spin-$\frac{1}{2}$ representation

SU(2)-symmetry: the strong force has this SU(2) symmetry



$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$

Similarly: $\boxed{I = 1 \quad \pi^+ = |1,1\rangle \quad \pi^0 = |1,0\rangle \quad \pi^- = |1,-1\rangle}$

Discovery of π, μ

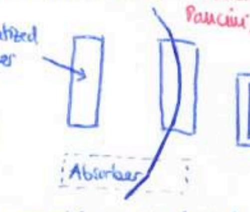
Anderson, Neddermeyer, Street, Stevenson

Cosmic rays: muon discovered → is it the meson? Conversi, Pancini, Piccioni

μ^+, μ^-: $m_{\mu^\pm} = 105\,\text{MeV}$

$\tau = 2{,}2\,\mu s$

$Br(\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu) \simeq 100\%$



- negative μ not absorbed as fast as it should be by Carbon
- no difference between positive/neg. particles

Powell, Fowler, Perkins, Lattes, Occhialini

Photographic emulsions of cosmic rays at high altitudes → recordings of π± tracks → "stars" following the capture of the meson: nucleus blasted apart

$m_\pi > m_\mu$ → it is a new particle!

π^+, π^-: $m_{\pi^\pm} = 140\,\text{MeV}$

$\tau = 2{,}6 \cdot 10^{-8}\,s$

$Br(\pi^+ \rightarrow \mu^+ + \nu_\mu) \simeq 100\%$

Studies of hadronic reactions produced by synchro-cyclotron

$p + C \longrightarrow \dots + \pi^0$

$\hookrightarrow \gamma\gamma$ (only occur above 200 MeV)

$$\underbrace{\tau_{\pi^0}(10^{-16}s)}_{\text{electromagnetic}} \ll \underbrace{\tau_{\pi^\pm}(10^{-8}s) \ll \tau_{\mu^\pm}(10^{-6}s)}_{\text{weak}}$$

π^0: $m_{\pi^0} = 135\,\text{MeV}$

$\tau = 8{,}5 \cdot 10^{-17}\,s$

$Br(\pi^0 \rightarrow \gamma\gamma) \simeq 99\%$

Nucleons and pions belong to separate isospin multiplets:

$$\boxed{\begin{array}{l} B = 0,\ I(J^P) = 1(0^-) : \ \pi^\pm, \pi^0 \\ B = 1,\ I(J^P) = \frac{1}{2}(\frac{1}{2}^+) : \ p, n \end{array}}$$

Gell-Mann: $Q = \frac{B}{2} + I_3$

Addition: consider
$\pi^+ + p \longrightarrow \pi^+ + p$
$\pi^- + p \longrightarrow \pi^- + p$
$\pi^- + p \longrightarrow \pi^0 + n$

$|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |\tfrac{3}{2}, \tfrac{3}{2}\rangle$

$|1,-1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\tfrac{3}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$|1,0\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}, -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$\mathcal{M}(\pi^+ + p \rightarrow \pi^+ + p) = \mathcal{M}_{\frac{3}{2}}$

$\mathcal{M}(\pi^- + p \rightarrow \pi^- + p) = \frac{1}{3}\mathcal{M}_{\frac{3}{2}} + \frac{2}{3}\mathcal{M}_{\frac{1}{2}}$

$\mathcal{M}(\pi^- + p \rightarrow \pi^0 + n) = \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}} - \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}$

Strangeness: more particles discovered in Wilson chamber and cloud chamber: $\boxed{K^\pm, K^0, \bar{K}^0, K_S^0, K_L^0, \Lambda^0, \Sigma^\pm, \Sigma^0}$

Strange particles could be copiously produced in strong interactions, however, they decayed with $\tau \simeq 10^{-9}s$ instead of $\tau \simeq 10^{-23}s$

Gell-Mann: $\boxed{Q = I_3 + \frac{B}{2} + \frac{S}{2}}$ — strangeness

$\pi, p, n : S = 0$
$K^+, K^0 : S = 1$
$K^-, \bar{K}^0, \Lambda, \Sigma^\pm, \Sigma^0 : S = -1$

→ strange particles always produced in pairs

Anti-Baryons

B conservation → baryons can only be produced in pairs

$\boxed{pCu \rightarrow \bar{p}X}$ in target config.: $E^* = \sqrt{2M_p^2 + 2M_pE_A} > 4M_p \rightarrow T_p > 6M_p$

Background: $pCu \rightarrow p\pi X$ → rejected via velocity: at $p = 1{,}2\,\text{GeV}$: $\beta_p \simeq 0{,}78$, $\beta_\pi \simeq 0{,}99$

→ velocity cut in Cherenkov counter + time-of-flight measurement

Anti-neutron: $p\bar{p} \rightarrow n\bar{n}$

$\hookrightarrow \bar{p} + \pi^+$, $\pi^+ \rightarrow \mu^+ \rightarrow e^+$, $\bar{p} \rightarrow$ star

Anti-Lambda: $\pi^- + p \rightarrow \Lambda^0 + \bar{\Lambda}^0 + n$

Bevatron

Spin and parity of pion

Deuteron $(p+n)$: $J^P = 1^+$, $I = 0$, $\varepsilon \simeq 2{,}2\,\text{MeV}$

The bound neutron cannot decay via $n \rightarrow p + e^- + \bar{\nu}_e$, since $2M_p + m_e > m_d$

Excited states of d do not exist

$\pi^+ + d \rightarrow pp$:

$$\frac{\sigma(p + p \rightarrow \pi^+ + d)}{\sigma(\pi^+ + d \rightarrow p + p)} = \frac{3}{4}(2S_\pi + 1)\frac{p_\pi^2}{p_p^2} \longrightarrow \boxed{S_\pi = 0}$$

→ π± is a pseudoscalar meson: $J^P = 0^-$

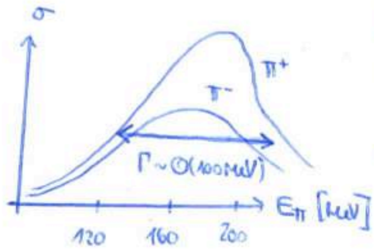
$\pi^- + d \rightarrow nn$: $\underbrace{\pi - d}_{J=1} \rightarrow \underbrace{n\,n}_{\text{have exchange symmetry}} \rightarrow \boxed{P_\pi = -1}$

fermions: $(-1)^{S+1}(-1)^{\ell} \overset{!}{=} (-1) \rightarrow \ell + S$ even

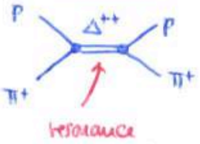
→ $S = 1$ and $\ell = 1$ only possibility to have $J = 1$

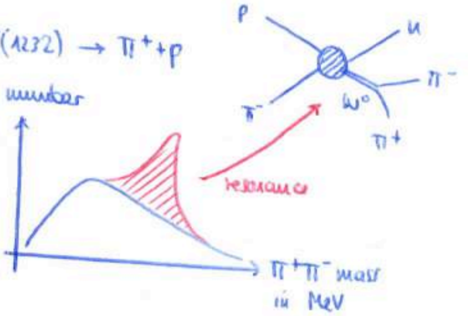
Resonances

$\sigma(\pi^+p) \gg \sigma(\pi^-p)$ for $E_\pi > 150$ MeV → resonance could be due to an unstable particle with $\tau \simeq 10^{-23}s \longrightarrow \Delta E \Delta t \simeq \hbar \rightarrow \Gamma \simeq \frac{\hbar}{\tau} = \mathcal{O}(100\,\text{MeV})$



Excited nucleon: $\pi^+ + p \longrightarrow \Delta^{++}(1232) \rightarrow \pi^+ + p$





→ in general, resonances turn up as peaks in $m^2 = (p_1+p_2)^2$

→ many new hadrons were found

Breit-Wigner

Stable particles: $\Psi(\vec{x},t) = e^{-iEt}\psi(\vec{x})$

Unstable particles: $\Psi(\vec{x},t) = e^{-m_Rt}e^{-\frac{\Gamma}{2}t}\psi(\vec{x})$

→ Fourier transform: $\tilde{\Psi}(E) = \frac{\psi(\vec{x})}{\sqrt{2\pi}}\frac{i}{(E-m_R)+i\frac{\Gamma}{2}} \longrightarrow \boxed{P(E) = \frac{\overset{\text{FWHM}}{\Gamma}}{(E-m_R)^2+\frac{\Gamma^2}{2}}} \longrightarrow \boxed{\sigma(s) \propto \frac{1}{(s-m_R^2)^2+m_R^2\Gamma^2}}$

→ mass of an unstable particle is not sharp → natural linewidth

Replace $m_R \rightarrow m_R - i\frac{\Gamma}{2} \rightarrow \ldots \boxed{\tilde{D}_F(p,m_R,\Gamma) = \frac{1}{p^2-m_R^2+im_R\Gamma}}$

propagator of a resonance

Decays

- strong: I, I_3 conserved
  hadronic final states
- electromagnetic: I, I_3 not conserved
  photons / charged lepton-antilepton pairs in final state
- weak: I, I_3 not conserved
  flavour may change
  neutrinos in final state

$\pi^0, \eta, \sigma_0 \longrightarrow$ decay via γ

$\pi^\pm, K^\pm, K^0, \Lambda, \Sigma^\pm, \Xi, \Omega^- \rightarrow$ decay via $W^\pm$

15. Hadrons (2/2)