1. Classical and Quantum Mechanics

Lagrangian Formalism: $S[q] = \int_{t_1}^{t_2} dt \, L(q^i(t), \dot{q}^i(t); t)$ $\quad L = T - V$

$$\delta S \overset{P.I.}{=} \int_{t_1}^{t_2} dt \, \delta q^i \underbrace{\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right)}_{\rightarrow \text{Euler-Lagrange}} + \underbrace{\int_{t_1}^{t_2} d\left(\delta q^i \frac{\partial L}{\partial \dot{q}^i} \right)}_{\text{boundary}: \; q_i(t_k) = \text{const. or } \frac{\partial L}{\partial \dot{q}^i}(t_k) = 0}$$

Hamiltonian Formalism: $p_i = \frac{\partial L}{\partial \dot{q}^i}$ $\quad H(\underbrace{q, p}_{\text{phase space}}; t) = p_i \dot{q}^i(q, p; t) - L(q, \dot{q}(q, p; t); t)$

$\rightarrow \delta H = \ldots \rightarrow \frac{\partial H}{\partial q^i} = -\dot{p}_i, \; \frac{\partial H}{\partial p_i} = \dot{q}_i$

Poisson brackets: $\{f, g\} := \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \rightarrow \frac{d}{dt} f(q, p; t) = \frac{\partial f}{\partial t} - \{H, f\}$

Example: Harmonic Oscillator:

$H = \frac{\vec{p}^2}{2m} + \frac{m\omega^2}{2} \vec{q}^2$

→ conveniently solved by

$\vec{a} = \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} + i\vec{p})$

$\vec{a}^* = \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} - i\vec{p})$

$\rightarrow \{f, g\} = -i \frac{\partial f}{\partial a^i} \frac{\partial g}{\partial a_i^*} + i \frac{\partial f}{\partial a_i^*} \frac{\partial g}{\partial a^i}$

$\rightarrow H = \omega \vec{a}^* \vec{a}$

Example: Harmonic Oscillator in QM:

$|\psi(t)\rangle = \int d\vec{q}^d \, \psi(t, \vec{q}) |\vec{q}\rangle$

$H = \frac{\vec{p}^2}{2m} + \frac{m\omega^2}{2} \vec{q}^2$ with $q^i = q^i, \; p_i = -i \frac{\partial}{\partial q^i}$

$\rightarrow [q^i, p_j] = i\hbar \delta^i_j$

Free particle ($\omega = 0$) solved by $|\vec{p}\rangle = \int d\vec{q}^d \, e^{\frac{i}{\hbar} \vec{p}\vec{q}} |\vec{q}\rangle$

Harmonic oscillator: $\vec{a} = \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} + i\vec{p})$

$\vec{a}^\dagger = \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} - i\vec{p})$

$\rightarrow H = \omega \vec{a}^\dagger \vec{a} + \frac{1}{2} d\hbar\omega$ $\quad [a^i, a_j^\dagger] = \hbar \delta^i_j$

$\rightarrow |\vec{n}\rangle = \left(\prod_{i=1}^{d} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!} \, \hbar^{n_i/2}} \right) |0\rangle$

$E = E_0 + \hbar\omega \sum_i n_i$

Quantum Mechanics:

$(q^i, p_i) \leftrightarrow |\psi\rangle \in \mathcal{V}$ $\quad i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$

$f(q, p; t) \leftrightarrow \hat{F}$ $\quad |\langle \phi | \psi \rangle|^2$ a probability

$\{f, g\} \leftrightarrow -i\hbar^{-1} [\hat{F}, \hat{G}]$ $\quad \rightarrow \langle \psi | \psi \rangle$ positive, normalizable, conserved

↳ $\frac{d}{dt} \langle \psi | \psi \rangle = 0$

↳ $H = H^\dagger$

Time Evolution: $|\psi(t_2)\rangle = e^{-iH(t_2 - t_1)} |\psi(t_1)\rangle$

Expectation value: $\langle \psi | F | \psi \rangle$

$\rightarrow \frac{d}{dt} \langle \psi | F | \psi \rangle = \langle \psi | \frac{dF}{dt} - \frac{1}{i\hbar} [H, F] | \psi \rangle$

Quantum Mechanics and Relativity:

Klein-Gordon: $\left(\underbrace{-\left(\frac{\partial}{\partial t}\right)^2}_{E^2} + \underbrace{\left(\frac{\partial}{\partial \vec{q}}\right)^2}_{-\vec{p}^2} \underbrace{- m^2}_{-m^2} \right) \psi(t, \vec{q}) = 0$

Issues:

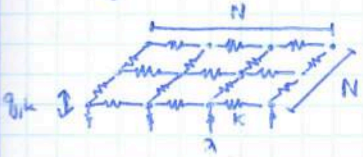
- ▶ Probabilistic properties: $\langle \psi | \psi \rangle$ conserved only for 1st order eq.
  $Q = \frac{i}{2m} (\langle \psi | \dot{\psi} \rangle - \langle \dot{\psi} | \psi \rangle)$ not positive def.
  (alternative Q not local)
- ▶ Causality: $\langle \vec{q}_2 | U(t_2, t_1) | \vec{q}_1 \rangle$ non-zero for space-like separations
  (there is however an exponential suppression)
- ▶ Negative Energy: solutions $|\vec{p}, -, t\rangle = \int d\vec{q}^d \, e^{i\vec{p}\vec{q} + ie(\vec{p})t} |\vec{q}\rangle$ exist
  ↳ neg. energy particles have not been observed, one may extract energy by making this particle (faster)
- ▶ Particle Creation: special relativity allows energy to be converted to mass
  ↳ in QM, however, particle numbers are fixed

Dirac: $\frac{\partial}{\partial t} \psi = \alpha^i \frac{\partial}{\partial q^i} \psi + m\beta\psi$

Issues:

- ▶ first order
- ▶ $\langle \psi | \psi \rangle$ conserved and pos. def.
  ↳ but $\langle \psi | \psi \rangle > 0$ requires Bose-Einstein statistics, while operators α^i imply Fermi statistics
- ▶ negative-energy solutions exist

Spring Lattice:



$$L = \frac{1}{2}\mu\sum_{i,j}\dot q_{i,j}^2 - \frac{1}{2}K\sum_{i,j}(q_{i-1,j}-q_{i,j})^2 - \frac{1}{2}K\sum_{i,j}(q_{i,j-1}-q_{i,j})^2 - \frac{1}{2}\lambda\sum_{i,j}q_{i,j}^2$$

→ Euler - Lagrange: $\mu\ddot q_{i,j} \begin{matrix} -K(q_{i-1,j}-2q_{i,j}+q_{i+1,j}) \\ -K(q_{i,j-1}-2q_{i,j}+q_{i,j+1}) \end{matrix} + \lambda q_{i,j} = 0$

→ Ansatz $q_{ij}(t) = \frac{1}{N^2}\sum_{k,l=1}^{N}\frac{\gamma_{k,l}}{\sqrt{2\mu\omega_{k,l}}}\exp\left(\frac{2\pi i}{N}(ki+lj)-i\omega_{kl}t\right) + c.c.$

→ Dispersion relation $\mu\omega_{k,l}^2 = \lambda + 4K\sin^2\frac{\pi k}{N} + 4K\sin^2\frac{\pi l}{N}$

Hamiltonian Formulation: $p_{i,j} = \frac{\partial L}{\partial\dot q_{i,j}} = \mu\dot q_{i,j}$, $\{f,g\} = \sum_{i,j}\left(\frac{\partial f}{\partial q_{i,j}}\frac{\partial g}{\partial p_{i,j}} - \frac{\partial f}{\partial p_{i,j}}\frac{\partial g}{\partial q_{i,j}}\right)$ with $\{q_{i,j}, p_{i,j}\} = \delta_{ik}\delta_{jl}$

Introducing $c_{k,l} = \frac{1}{\sqrt{2\mu\omega_{k,l}}}\sum_{i,j=1}^{N}\exp\left(-\frac{2\pi i}{N}(ki+lj)\right)(\mu\omega_{k,l}q_{i,j}+ip_{i,j}) \longrightarrow H = \frac{1}{N^2}\sum_{k,l}\omega_{k,l}c_{k,l}^*c_{k,l}$

with $\{c_{ij}, c_{kl}^*\} = -iN^2\delta_{i,k}\delta_{j,l} \longrightarrow \dot c_{kl} = -\{H, c_{kl}\} = -i\omega_{k,l}c_{k,l}$

Continuum Limit: let $N\to\infty$, $L\to\infty$, $r = \frac{L}{N}\to 0$, $x = i\frac{L}{N}$, $q_{i,j} = \varphi(\vec x)$, generalized to d spatial dimensions

Using $\sum_{i=1}^{N}\to\frac{1}{r}\int dx$, $q_i - q_{i-1}\to r(\partial\varphi)$, $\mu = r^d\bar\mu$, $K = r^{d-2}\bar K$, $\lambda = r^d\bar\lambda$ and $\varphi = \bar K^{-\frac{1}{2}}\phi$:

$$L[\phi,\dot\phi](t) = \int d\vec x^d\left(\frac{1}{2}\underbrace{\bar\mu\bar K^{-1}}_{c^{-2}=1}\dot\phi^2 - \frac{1}{2}(\vec\partial\phi)^2 - \frac{1}{2}\underbrace{\bar\lambda\bar K^{-1}}_{m^2}\phi^2\right) \longrightarrow \text{E.L.}: \; -\ddot\phi + \vec\partial^2\phi - m^2\phi = 0 \quad \text{(Klein-Gordon)}$$

2. Classical Free Scalar Field

Plane Wave Solutions:

$$\boxed{\phi(\vec x,t) = \int\frac{d\vec p^d}{(2\pi)^d 2\epsilon(\vec p)}\alpha(\vec p)e^{i\vec p\vec x - i\epsilon(\vec p)t} + c.c.}$$

agreeing with discrete solution for

$p = \frac{2\pi k}{L}$ $\sum_{k=1}^{N}\to\frac{L}{2\pi}\int dp$ $\gamma_{k..}\to\frac{\alpha(\vec p)}{\sqrt{2\epsilon(\vec p)r^d}}$

this solution is found

the same solution is found

Relativistic Covariance:

Klein - Gordon: $\boxed{-\partial^\mu\partial_\mu\phi + m^2\phi = 0}$

$\boxed{\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2}$

Momentum space: $(p^2+m^2)\phi(p) = 0$

$\longrightarrow \boxed{\phi(p) = 2\pi\,\delta(p^2+m^2)\left(\Theta(p^0)\alpha(\vec p) + \Theta(-p^0)\alpha^*(-\vec p)\right)}$

(since $\phi(x)^* = \phi(x) \leftrightarrow \phi(p)^* = \phi(-p)$)

$$\int\frac{dp^0}{(2\pi)^D}2\pi\,\delta(p^2+m^2)\Theta(p^0)f(p) = \int\frac{d\vec p^d}{(2\pi)^d 2\epsilon(\vec p)}f(\epsilon(\vec p),\vec p)$$

$\Lsh$ $\delta(f(x)) = \sum_{x_0}\frac{\delta(x-x_0)}{|f'(x_0)|}$

Hamiltonian Field Theory: $\pi(\vec x,t) = \dot\phi(\vec x,t) \longrightarrow H[\phi,\pi] = \int d\vec x^d\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right)$

$\{f,g\} = \int d\vec x^d\left(\frac{\delta f}{\delta\phi(\vec x)}\frac{\delta g}{\delta\pi(\vec x)} - \frac{\delta f}{\delta\pi(\vec x)}\frac{\delta g}{\delta\phi(\vec x)}\right) \longrightarrow \{\phi(\vec x), \pi(\vec y)\} = \delta^d(\vec x-\vec y)$

$$\boxed{\begin{aligned} a(\vec p) &= \int d\vec x^d\exp(-i\vec p\vec x)\left(\epsilon(\vec p)\phi(\vec x) + i\pi(\vec x)\right) \\ a^*(\vec p) &= \int d\vec x^d\exp(+i\vec p\vec x)\left(\epsilon(\vec p)\phi(\vec x) - i\pi(\vec x)\right)\end{aligned}} \longleftrightarrow \boxed{\begin{aligned} \phi(\vec x) &= \int\frac{d\vec p^d}{(2\pi)^d 2\epsilon(\vec p)}\left(a(\vec p)\exp(+i\vec p\vec x) + c.c.\right) \\ \pi(\vec x) &= -\frac{i}{2}\int\frac{d\vec p^d}{(2\pi)^d}\left(a(\vec p)\exp(+i\vec p\vec x) - c.c.\right)\end{aligned}}$$

$\{f,g\} = -i(2\pi)^d\int d\vec p^d\,2\epsilon(\vec p)\left(\frac{\delta f}{\delta a(\vec p)}\frac{\delta g}{\delta a^*(\vec p)} - \frac{\delta f}{\delta a^*(\vec p)}\frac{\delta g}{\delta a(\vec p)}\right) \longrightarrow \{a(\vec p), a^*(\vec q)\} = -i\,2\epsilon(\vec p)(2\pi)^d\delta^d(\vec p-\vec q)$

$$H = \int\frac{d\vec p^d}{(2\pi)^d 2\epsilon(\vec p)}\epsilon(\vec p)a^*(\vec p)a(\vec p) \longrightarrow \left.\begin{aligned}\dot a(\vec p) &= -i\epsilon(\vec p)a(\vec p) \\ \dot a^*(\vec p) &= i\epsilon(\vec p)a^*(\vec p)\end{aligned}\right\} \begin{aligned} a(\vec p,t) &= \alpha(\vec p)\exp(-i\epsilon(\vec p)t) \\ a^*(\vec p,t) &= \alpha^*(\vec p)\exp(i\epsilon(\vec p)t)\end{aligned}$$

Quantization: Direct Approach:

$$\hat{\phi}(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle, \quad \hat{\pi}(\vec{x})|\phi\rangle = i\frac{\delta}{\delta\phi(\vec{x})}|\phi\rangle$$

Use $\{\phi(\vec{x}), \pi(\vec{x})\} = \delta^d(\vec{x}-\vec{y}) \rightarrow [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^d(\vec{x}-\vec{y})$

$$\rightarrow |0\rangle = \int D\phi \ \exp\left(-\tfrac{1}{2}\int d\vec{x}^d d\vec{y}^d \, \Omega(\vec{x},\vec{y})\phi(\vec{x})\phi(\vec{y})\right) |\phi\rangle$$

inconvenient

Quantization: Ladder Operators

Quantize ladder operators $[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = 2\epsilon(\vec{p})(2\pi)^d \delta^d(\vec{p}-\vec{q}) \rightarrow$

$$H = \int d\vec{x}^d \left(\tfrac{1}{2}\overset{\text{functional of } a, a^\dagger}{\hat{\pi}^2} + \tfrac{1}{2}(\vec{\partial}\hat{\phi})^2 + \tfrac{1}{2}m^2\hat{\phi}^2\right)$$

$$= \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} \epsilon(\vec{p}) a^\dagger(\vec{p})a(\vec{p}) + \underbrace{\tfrac{1}{2}\overset{\text{UV div.}}{\int d\vec{p}^d \epsilon(\vec{p})} \overset{\text{IR div.}}{\delta^d(\vec{p}-\vec{p})}}_{E_0}$$

E_0 diverges:

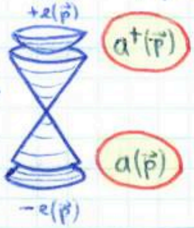
- ▶ $\delta^d(0)$ ill defined, since we consider an infinite volume (IR divergence): $\delta^d(\vec{p}-\vec{p}) \sim V \rightarrow \infty$
  - → put system in finite box or consider energy density
- ▶ $\frac{1}{2}\int d^3\vec{p} \ \epsilon(\vec{p})$ diverges, since we consider infinitely many harmonic oscillators (UV divergence)
  - → introduce momentum cutoff $\frac{1}{2}\int_0^{\Lambda} d^3\vec{p}\,\epsilon(\vec{p})$

Here we may simply drop E_0, since there is no meaning to absolute energies

$$\rightarrow \boxed{H_{ren} = \frac{1}{2}\int\frac{d\vec{p}^d}{(2\pi)^d} a^\dagger(\vec{p})a(\vec{p})} = N[H]$$

Fock Space: $|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$ has energy $E = +\epsilon(\vec{p})$. $a(\vec{p})|0\rangle$ would have negative energy, but it does not exist → problem with negative-energy solutions resolved!

Note: ϕ, π create/annihil. particles at the same time → superposition of states with diff. particle numb.



3. Scalar Field Quantisation

Normalization: $\langle\vec{p}|\vec{p}\rangle = 2\epsilon(\vec{p})(2\pi)^d\delta^d(\vec{p}-\vec{p}) = \infty$,

but $|\psi\rangle = \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})}\psi(\vec{p})|\vec{p}\rangle$ has a finite normalization

Conservation laws:

$$P^\mu = \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} \overset{(\epsilon(\vec{p}), \vec{p})}{p^\mu(\vec{p})} a^\dagger(\vec{p})a(\vec{p}) \quad \text{conserved}$$

$$N = \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} a^\dagger(\vec{p})a(\vec{p}) \quad \text{conserved}$$

In fact, $[P_\mu, n(\vec{p})] = [N, n(\vec{p})] = 0$

→ any operator composed of $n(\vec{p})$ is conserved

Multi-Particle States:

$$\boxed{|\vec{p}_1, \dots, \vec{p}_n\rangle = a^\dagger(\vec{p}_1)\dots a^\dagger(\vec{p}_n)|0\rangle} \rightarrow E = \sum_{k=1}^{n}\epsilon(\vec{p}_k)$$

↳ follows Bose-Einstein statistics since $[a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0$

$[H, a^\dagger(\vec{p})] = \epsilon(\vec{p})a^\dagger(\vec{p})$

Fock space: $V_{Fock} = \underset{|0\rangle}{V_0} \oplus \underset{|\vec{p}\rangle}{V_1} \oplus V_2 \oplus \dots$ $\qquad V_n = (V_1)^{\otimes_s n}$ (symmetric)

bounded from above in low-energy limit

Complex Scalar Field:

$$\boxed{\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi} \quad \left(\phi(x) = \tfrac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\right)$$

$\pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*$ and $[\phi(\vec{x}), \pi(\vec{y})^\dagger] = i\delta^d(\vec{x}-\vec{y})$

$$\phi(x) = \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} b(\vec{p})\exp(ipx) + \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} a^\dagger(\vec{p})\exp(-ipx)$$

$$\phi^\dagger(x) = \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} a(\vec{p})\exp(ipx) + \int\frac{d\vec{p}^d}{(2\pi)^d 2\epsilon(\vec{p})} b^\dagger(\vec{p})\exp(-ipx)$$

$$\boxed{[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = 2\epsilon(\vec{p})(2\pi)^d\delta^d(\vec{p}-\vec{q})}$$

$$\boxed{H_{ren} = \frac{1}{2}\int\frac{d\vec{p}^d}{(2\pi)^d}\left(a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})\right)}$$

Correlators: Heisenberg picture: $F_H(t) = \exp(iH(t-t_0))\, F_S(t)\, \exp(-iH(t-t_0))$

→ time-dependence of fields shifted to time-dependence of operators

$$\rightarrow \phi(x) = \int \frac{d\vec{p}^{\,d}}{(2\pi)^d 2e(\vec{p})} \left(e^{ipx} a(\vec{p}) + e^{-ipx} a^\dagger(\vec{p})\right) \quad \text{with } p^0 = e(\vec{p}) \text{ implied}$$

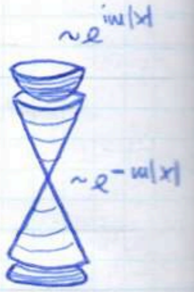
→ has complete space-time dependence, $\pi = \dot{\phi}$

$$\rightarrow \boxed{\Delta_+(y,x) = i\langle 0|\phi(y)\phi(x)|0\rangle = i\int \frac{d\vec{p}^{\,d}}{(2\pi)^d 2e(\vec{p})} \exp(ip(y-x))} \rightarrow \Delta_+(p) = 2\pi i\, \delta(p^2+m^2)\Theta(p^0)$$

Explicit expression: $\Delta_+(x^2) = m^{d-1} F(m^2x^2) \longrightarrow 4r F''(r) + 2(d+1)F'(r) - F(r) = 0$

↑ Poincaré symmetry, dimensional analysis

$$\rightarrow F_\pm(r) = r^{-(d-1)/4} J_{\pm(d-1)/2}(i\sqrt{r})$$



↙ Hankel fct.

Future: $\Delta_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} H_{(d-1)/2}\left(m\sqrt{-(y-x)^2}\right) \overset{t\to\pm\infty}{\sim} e^{-imt}$ for $|t| = \sqrt{-x^2} \to \infty$

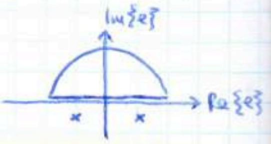
Space-like: $\Delta_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} K_{(d-1)/2}\left(m\sqrt{(y-x)^2}\right) \sim e^{-mr}$ for $r = \sqrt{x^2} \to \infty$

↑ modified Bessel fct.

Unequal-Time Correlator: $\boxed{\Delta(x-y) = i[\phi(x), \phi(y)]} \rightarrow \langle 0|\Delta(x-y)|0\rangle = \boxed{\Delta_+(x-y) - \Delta_+(y-x)}$

- Δ_+ symmetric for space-like separations → $\Delta(x-y)$ vanishes → causality preserved
  - ↳ also follows from invariance and the fact that the equal-time commutator is a delta-function
- $\Delta(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} J_{(d-1)/2}\left(m\sqrt{-(y-x)^2}\right)$ for time-like separations
- equal-time commutator can be recovered

Green's Function: A solution to $-\partial^2\phi(x) + m^2\phi(x) = \rho(x)$ can be constructed from $\phi(x) = \phi_i(x) + \Delta\phi(x)$, where $\phi_i(x)$ is a homogeneous solution and $\Delta\phi(x) = \int dy^{d+1} G_R(x-y)\rho(y)$



Green's function: $\boxed{-\partial^2 G_R(x) + m^2 G_R(x) = \delta^{d+1}(x)}$ $\boxed{G_R(x) = 0,\ x^0 < 0}$

→ Momentum space: $(p^2+m^2)G(p) = 1 \longrightarrow G(p) = \frac{1}{p^2+m^2} \longrightarrow$ handling of the poles: $G_R(p) = \frac{1}{p^2+m^2-ip^0\varepsilon}$

$$\rightarrow \boxed{G_R(x)} = \int \frac{dp^4}{(2\pi)^4} e^{ipx} \frac{1}{p^2+m^2-ip^0\varepsilon} = \boxed{\Theta(t)\Delta(x)} \rightarrow \text{indeed: } (-\partial^2+m^2)G_R(x) = \delta(t)\dot{\Delta}(x) = \delta^{d+1}(x)$$

$$\rightarrow \Delta\phi \underset{\uparrow}{=} \int dy^{d+1} \Delta(x-y)\rho(y) = \int \frac{d\vec{p}^{\,d}}{(2\pi)^d 2e(\vec{p})} \left(ie^{ipx}\rho(p) - ie^{-ipx}\rho^*(p)\right) \longrightarrow \boxed{a_f(\vec{p}) = a_i(\vec{p}) + i\rho(e(\vec{p}), \vec{p})}$$

x far in the future

4. Symmetries

Noether's Theorem: Every continuous global symmetry of the action leads to a conserved current and thus a conserved charge for solutions of the equations of motion

$\delta\mathcal{L} \overset{E.L}{=} \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right)$ and $\delta\mathcal{L} = \delta\alpha\, \partial_\mu J_0^\mu$ for it to leave $\delta S = 0$

$\longrightarrow \boxed{J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} - J_0^\mu}$ is conserved: $\partial_\mu J^\mu = 0$ and $\boxed{Q(t) = \int d\vec{x}^d\, J^0(t,\vec{x})}$ is a conserved charge: $\frac{d}{dt}Q(t) = 0$.

$\longrightarrow$ Q generates an infinitesimal symmetry transformation via $\boxed{\{Q,F\} = -\frac{\delta F}{\delta\alpha}}$

Global Gauge Symmetry: The transformation $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ leaves the Lagrangian of the complex scalar field invariant $\longrightarrow \delta\phi = i\alpha\phi(x) \longrightarrow \boxed{J^\mu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} = \boxed{-i(\partial^\mu\phi^*\phi - \partial^\mu\phi\phi^*)}$

$\longrightarrow \boxed{Q = \int\frac{d\vec{p}^d}{(2\pi)^d 2\omega(\vec{p})}\left(a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p})\right)}$ Hermitian

unitary, since Q Hermitian

$[Q, a(\vec{p})] = -a(\vec{p})$ $\quad [Q, b(\vec{p})] = +b^\dagger(\vec{p})$ $\quad [Q, \phi(x)] = +\phi(x)$

$[Q, a^\dagger(\vec{p})] = +a^\dagger(\vec{p})$ $\quad [Q, b^\dagger(\vec{p})] = -b(\vec{p})$ $\quad [Q, \phi^\dagger(x)] = -\phi(x)$

$\Big\}\ U(\alpha) = \exp(i\alpha Q)$ with $U(\alpha)\phi(x)U(\alpha)^{-1} = e^{i\alpha}\phi(x)$, $U(\alpha)\phi^\dagger(x)U(\alpha)^{-1} = e^{-i\alpha}\phi^\dagger(x)$

$\rightarrow$ Q is the generator

Space-time translations: $\boxed{(x')^\mu = x^\mu + a^\mu}$

We demand $\phi'(x') = \phi(x) \rightarrow \delta\phi(x) = -\delta a^\mu\partial_\mu\phi(x)$

We require $\frac{\partial\mathcal{L}}{\partial x^\mu} = 0$ for it to be a symmetry

$\longrightarrow \quad \delta\mathcal{L} = -\delta a^\mu\,\partial_\mu(\mathcal{L}(\phi, \partial_\mu\phi))$

$\longrightarrow \boxed{T^{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L}}$ conserved

energy-momentum tensor

$P^\mu = \int d\vec{x}^d\, T^{0\mu} = \ldots = \int\frac{d\vec{p}^d}{(2\pi)^d 2\omega(\vec{p})}\, p^\mu a^\dagger(\vec{p})a(\vec{p})$

$\longrightarrow [P^\mu, \phi(x)] = i\partial^\mu\phi(x)$ and

$U(a) = \exp(ia^\mu P_\mu)$, $U(a)\phi(x)U(a)^{-1} = \phi(x-a)$

Poincaré UIRs:

For massive particles: $\boxed{|\vec{p}, j_3\rangle_{(m,\pm,j)}}$

For massless particles: $\boxed{|\vec{p}\rangle_{(0,\pm,h)}}$

Space-time rotations: $\boxed{(x')^\mu = (\Lambda^{-1})^\mu{}_\nu\, x^\nu}$

with $\boxed{\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu}$ (Λ conserves the Minkowski metr.)

$\Lambda^\mu{}_\nu = \exp(\omega)^\mu{}_\nu$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$

- spatial rot.: $\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} = \exp\begin{pmatrix}0 & -\varphi\\ \varphi & 0\end{pmatrix}$
- boosts: $\begin{pmatrix}\cosh\vartheta & \sinh\vartheta\\ \sinh\vartheta & \cosh\vartheta\end{pmatrix} = \exp\begin{pmatrix}0 & \vartheta\\ \vartheta & 0\end{pmatrix}$

$\phi'(x) = \phi(\Lambda x) \rightarrow \delta\phi = \delta\omega^\mu{}_\nu\, x^\nu\partial_\mu\phi$

and we require $\delta\mathcal{L} = \delta\omega^\mu{}_\nu\, x^\nu\partial_\mu\mathcal{L} = \delta\omega_{\mu\nu}\,\partial^\mu(x^\nu\mathcal{L})$

$\longrightarrow \boxed{T^{\mu\nu} = T^{\nu\mu}}$

$\longrightarrow \boxed{J^{\mu,\rho\sigma} = -T^{\mu\rho}x^\sigma + T^{\mu\sigma}x^\rho = -J^{\mu,\sigma\rho}}$

$J^m = -\frac{1}{2}\varepsilon^{mkl}M^{kl}$, $K^m = M^{m0}$

$M^{\mu\nu} = \int d\vec{x}^d\, J^{0,\mu\nu} = i\int\frac{d\vec{p}^d}{(2\pi)^d 2\omega(\vec{p})}\left(p^\mu\partial^\nu a^\dagger(\vec{p})a(\vec{p}) - p^\nu\partial^\mu a^\dagger(\vec{p})a(\vec{p})\right) \longrightarrow U(\omega) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$

Poincaré Symmetry:

Lie group / Lie algebra

Representation: $R: X \rightarrow \mathrm{End}(V)$ with $R(a)R(b) = R(ab)$

$[R(a), R(b)] = R([a,b])$

$u(N)$: Hermitian, $su(N)$: Hermitian, traceless, $so(N)$: anti-symmetric

$so(N,M)$: antisymmetric w.r.t given signature

Double cover: $SO(N,M)$ has double cover $Spin(N,M)$

$\rightarrow$ in the latter, a rotation by 2π is a non-trivial element

Poincaré algebra:

$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\nu\rho}M^{\mu\sigma} - i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\sigma}M^{\mu\rho} + i\eta^{\mu\sigma}M^{\nu\rho}}$

$\boxed{[M^{\mu\nu}, P^\rho] = i\eta^{\nu\rho}P^\mu - i\eta^{\mu\rho}P^\nu}$

$\boxed{[P^\mu, P^\nu] = 0}$ $\quad \boxed{U(\omega, a) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu\right)}$

Discrete Symmetries:

$$P\phi(t,\vec{x})P^{-1} = \eta_P \phi(t,-\vec{x})$$
$$P\phi^\dagger(t,\vec{x})P^{-1} = \eta_P^* \phi^\dagger(t,-\vec{x})$$
$$\rightarrow P^2 = 1 \longrightarrow \eta_P = \pm 1$$

$$Pa^\dagger(\vec{p})P^{-1} = \eta_P a^\dagger(-\vec{p})$$
$$Pa(\vec{p})P^{-1} = \eta_P a(-\vec{p})$$
$$Pb(\vec{p})P^{-1} = \eta_P b(-\vec{p})$$
$$Pb^\dagger(\vec{p})P^{-1} = \eta_P b^\dagger(-\vec{p})$$

⚠ in the presence of internal symmetries $\phi \to e^{i\alpha}\phi$, $\phi \to -\phi$, η_c becomes ambiguous

↳ one could define $P' = PC$

$$T\phi(t,\vec{x})T^{1} = \eta_T \phi(-t,\vec{x})$$
$$T\phi^\dagger(t,\vec{x})T^{-1} = \eta_T^* \phi^\dagger(-t,\vec{x})$$
$\longrightarrow$ implemented by anti-linear $\bar{T}$, $\bar{T}^2 = 1$

$\rightarrow |\eta_{\bar{T}}|^2 = 1$

$$\bar{T}a^\dagger(\vec{p})\bar{T}^{-1} = \eta_{\bar{T}} a^\dagger(-\vec{p})$$
$$\bar{T}a(\vec{p})\bar{T}^{-1} = \eta_{\bar{T}}^* a(-\vec{p})$$
$$\bar{T}b(\vec{p})\bar{T}^{-1} = \eta_{\bar{T}} b(-\vec{p})$$
$$\bar{T}b^\dagger(\vec{p})\bar{T}^{-1} = \eta_{\bar{T}}^* b^\dagger(-\vec{p})$$

$$C\phi(x)C^{-1} = \eta_c \phi^\dagger(x)$$
$$C\phi^\dagger(x)C^{-1} = \eta_c^* \phi(x)$$
$$\longrightarrow C^2 = 1 \rightarrow |\eta_c|^2 = 1$$

$$Ca^\dagger(\vec{p})C^{-1} = \eta_c b^\dagger(\vec{p})$$
$$Ca(\vec{p})C^{-1} = \eta_c^* b(\vec{p})$$
$$Cb(\vec{p})C^{-1} = \eta_c a(\vec{p})$$
$$Cb^\dagger(\vec{p})C^{-1} = \eta_c^* a^\dagger(\vec{p})$$

Note: a discrete transformation is a symmetry if it commutes with the Hamiltonian

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Poincaré Representations: We'd like to find the unitary irreducible representations (UIRs) of the Poincaré group.

$P^2 = P^\mu P_\mu$ commutes with $P^\mu, M^{\mu\nu}$ $\longrightarrow$ P^2 must be represented by a unique number on an irreducible representation: $P^2 = -m^2$ (for unitary representations, P^μ must be Hermitian $\rightarrow m^2$ real)

P^μ span an Abelian ideal (subalgebra: $[M,P] \sim P$) $\longrightarrow$ we may choose simultaneous eigenstates $|p\rangle$ of all generators P^μ as basis vectors for the representation space: $P^\mu|p\rangle = p^\mu|p\rangle$

$$P^2 = -m^2 \longrightarrow p^\mu = (\pm e(\vec{p}), \vec{p})$$

$|\vec{p}\rangle_{m,+} \longrightarrow$ irreducible rep. on $|\vec{p}\rangle_{m,+} = |+e_m(\vec{p}), \vec{p}\rangle$

note: $|\vec{p}\rangle_{m,-} \sim \langle +e_m(\vec{p}), \vec{p}|^\dagger$

$|\vec{p}\rangle_{m,-} \longrightarrow$ irreducible rep. on $|\vec{p}\rangle_{m,-} = |-e_m(\vec{p}), \vec{p}\rangle$

Spin: $(m, \vec{0})$ has a $SO(d)$ or $Spin(d)$ symmetry ("little group" of p^μ)

$d = 3 \longrightarrow$ we get representations with $j = \frac{1}{2}\mathbb{N}_0$: $|-j\rangle_j, |-j+1\rangle_j, \dots, |j\rangle_j$

Helicity: for massless representations, we have a $U(1) = Spin(2)$ symmetry

Dirac Equation: $(\gamma^\mu\partial_\mu - m)\Psi = 0 \rightarrow (\gamma^\mu\partial_\mu + m)(\gamma^\mu\partial_\mu - m)\Psi = 0$ for $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

▸ $d = 3\ (+++)$: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \sigma^j\sigma^k = \delta^{jk} + i\varepsilon^{jkl}\sigma^l \rightarrow \{\sigma^i, \sigma^j\} = 2\delta^{ij}$

▸ $d = 4\ (-+++)$: Clifford algebra can be realized as 4×4-matrices

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$ with $\sigma^0 = -\bar\sigma^0 = 1\!\!1$, $\bar\sigma^k = \sigma^k$ $\rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Solutions: $\Pi^\pm = \frac{1}{2m}(m \mp i p\!\!\!/)$ → Dirac eq. becomes $\Pi^+\Psi = 0$, $\Pi^-\Pi^+\Psi = \frac{1}{4m^2}(m^2 + p^2)\Psi$ → we need $p = -m^2$

→ On mass shell: Π^+, Π^- form a complete set of orthogonal projectors

→ Solutions $u_\alpha(\vec p),\ \alpha = \pm,\ (i p\!\!\!/ - m)u_\alpha(\vec p) = 0$

$v_\alpha(\vec p),\ \alpha = \pm,\ (i p\!\!\!/ + m)v_\alpha(\vec p) = 0$

$$\rightarrow \Psi(x) = \int \frac{d\vec p^3}{(2\pi)^3 2\omega(\vec p)} \sum_{\gamma=\pm} \left(e^{ipx} u_\gamma(\vec p) b_\gamma(\vec p) + e^{-ipx} v_\gamma(\vec p) a^\dagger_\gamma(\vec p) \right)$$

→ Weyl: $u(\vec p) = \frac{1}{2m}\begin{pmatrix} mK \\ i\omega(\vec p)K + i\vec p\vec\sigma K \end{pmatrix}$, $v(\vec p) = \frac{1}{2m}\begin{pmatrix} mK \\ -i\omega(\vec p)K - i\vec p\vec\sigma K \end{pmatrix}$ with $K = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Poincaré Symmetry: $x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$

→ $\Psi'(x) = S(\omega)\Psi(x)$ Spinor transformation

$0 \overset{!}{=} (\gamma^\mu\partial_\mu - m)\Psi'(x) = S(\Lambda^\mu{}_\nu S^{-1}\gamma^\nu S - \gamma^\mu)$

$(\partial_\mu\Psi) \rightarrow (\Lambda^{-1})^\mu{}_\nu\, S\gamma^\nu S^{-1} \overset{!}{=} \gamma^\mu$

→ $[\delta S, \gamma^\mu] - \delta\omega^\mu{}_\nu\gamma^\nu = 0$

satisfied by $\delta S = -\frac{1}{4}\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu$

Spin(N) represent: 4π periodic

→ $S(\omega) = \exp\left(-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\right)$

→ new representation of the Lorentz group on spinors: $M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

5. Free Spinor Field

Chiral Representation:

$M^{\mu\nu} = \frac{i}{4}\begin{pmatrix} \boxed{\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu} & 0 \\ 0 & \boxed{\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu} \end{pmatrix}$

→ block-diagonal

↳ two independent representations $M_L^{\mu\nu}, M_R^{\mu\nu}$ on $\Psi = (\Psi_L, \Psi_R)$

The mass term in the Dirac eq. mixes the two → chirality is not conserved in free propagation (for massive part.)

In other representations: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ with

$\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^2 = 1\!\!1$ $\quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in Weyl

$\gamma_{R,L} = \frac{1}{2}(1 \pm \gamma^5)$ project out chiral subspaces

Spin Statistics: $\mathcal{L} = \bar\Psi(\gamma^\mu\partial_\mu - m)\Psi$ → E.L. yields normal and c.c. Dirac equation

$\mathcal{L}^\dagger = \mathcal{L} - \underbrace{\partial_\mu(\bar\Psi\gamma^\mu\Psi)}_{\text{topological term}}$ almost real

Hamiltonian: $\pi = \frac{\partial\mathcal{L}}{\partial\dot\Psi^\dagger} = 0$, $\pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot\Psi} = i\Psi^\dagger$

→ $\Psi, \Psi^\dagger$ are canonically conjugate, π and $\pi^\dagger$ are not needed (phase space equals position space)

We don't follow the canonical quantization procedure (complications!), instead:

$T^{\mu\nu} = -\bar\Psi\gamma^\mu\partial^\nu\Psi + \eta^{\mu\nu}\mathcal{L}$ with $T^{\mu\nu} - T^{\nu\mu} = \partial_\rho K^{\rho\mu\nu}$ almost symmetric

$$\rightarrow H = \int d\vec x^3\, T^{00} = \int d\vec x^3\, \bar\Psi(-\vec\gamma\vec\partial + m)\Psi$$

A naive quantization treatment leads to $H[\Psi_c] = -H[\Psi] < 0$

(for the boson field we get $H[\phi_c] = +H[\phi]$)

→ this can be resolved by requiring $\Psi^a\Psi_b^\dagger = -\Psi_b^\dagger\Psi^a$

Half-integer spin: $\{\Psi, \Psi^\dagger\} \sim \mathcal{L}$ → Fermi-Dirac statistics

Integer spin: $[\phi, \phi^\dagger] \sim \mathcal{L}$ → Bose-Einstein statistics

Discrete Symmetries:

Parity: $\Psi'(t, -\vec x) = \gamma_P\Psi(t, \vec x)$, where

$\Lambda^\mu{}_\nu\gamma_P\gamma^\nu\gamma_P^{-1} \overset{!}{=} \gamma^\mu \rightarrow \gamma_P = -i\gamma^0$

→ γ_P interchanges the chiralities

Time Reversal: $\Psi'(-t, \vec x) = \gamma_T\Psi(t, \vec x)$

$\Lambda^\mu{}_\nu\gamma_T(\gamma^\nu)^*\gamma_T^{-1} \overset{!}{=} \gamma^\mu \rightarrow \gamma_T = i\gamma^0\gamma^2$

Charge Conj.: $\Psi'(x) = \gamma_C\Psi^{\dagger T}$

$(\gamma^\mu\partial_\mu - m)\Psi' \overset{!}{=} 0 \rightarrow \gamma^\mu = \gamma_C(\gamma^\mu)^*\gamma_C^{-1}$

→ $\gamma_C = i\gamma^0\gamma^1\gamma^3$

CPT: $\Psi'(x) = \gamma_T\gamma_P\gamma_C\Psi^{\dagger T}(-x)$

with $\gamma_T\gamma_P\gamma_C = \gamma^5$

→ all reasonable QFTs are invariant under CPT transformations

Hermitian Conjugate: $\gamma_+(\gamma^\mu)^\dagger\gamma_+^{-1} = -\gamma^\mu$

$\bar\Psi = \Psi^\dagger\gamma_+^{-1}$ $\quad \bar X = \gamma_+ X^\dagger\gamma_+^{-1}$

$\gamma_+ = -i\gamma^0 \quad (\gamma_+^\dagger = \gamma_+,\ \gamma_+^* = -\gamma_+)$

Quantization: $\{F,G\} = i\int d\vec{x}^3 \left(\frac{\delta F}{\delta\psi^a(x)}\frac{\delta G}{\delta\psi^\dagger_a(x)} + \dots\right)$

$\rightarrow \{\psi^a(x),\psi^\dagger_b(y)\} = i\delta^a_b\,\delta^3(\vec{x}-\vec{y})$

$\rightarrow$ Can. quantization: $\{\psi^a(\vec{x}),\psi^\dagger_b(\vec{y})\} = \delta^a_b\,\delta^3(\vec{x}-\vec{y})$

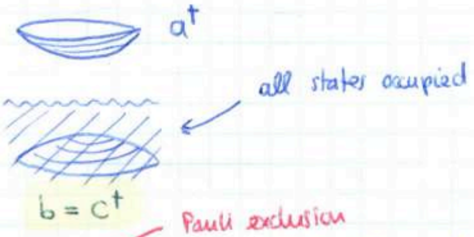
$$\rightarrow \psi(x) = \int\frac{d\vec{p}^3}{(2\pi)^3 2\omega(\vec{p})}\left(e^{ipx}u_\alpha(\vec{p})b_\alpha(\vec{p}) + e^{-ipx}v_\alpha(\vec{p})a^\dagger_\alpha(\vec{p})\right)$$

Using the completeness relations: $u_\alpha(\vec{p})\bar{u}_\alpha(\vec{p}) = ip\gamma + m$

$v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) = ip\gamma - m$

$$\rightarrow \{a_\alpha(\vec{p}),a^\dagger_\beta(\vec{q})\} = \{b_\alpha(\vec{p}),b^\dagger_\beta(\vec{q})\} = \delta_{\alpha\beta}\,2\omega(\vec{p})\,(2\pi)^3\delta^3(\vec{p}-\vec{q})$$

Dirac Sea:



Pauli exclusion

All states are occupied → vacuum annihilated by $c^\dagger$, since $(c^\dagger)^2 = \frac{1}{2}\{c^\dagger,c^\dagger\} = 0$, hole created by c corresponding to a negative energy particle

$\rightarrow$ Nowadays: $b = c^\dagger$, works for bosons as well, there are anti-particles

Correlators: $\Delta^{D\,a}_{\pm\;b}(x-y) = i\langle 0|\psi^a(x)\bar{\psi}_b(y)|0\rangle$ $\quad -i\langle 0|\bar{\psi}_b(y)\psi^a(x)|0\rangle$

$\rightarrow \Delta^D_\pm(x) = i\int\frac{d\vec{p}^3}{(2\pi)^3 2\omega(\vec{p})}\,e^{ipx}\,(-ip\gamma+m)^a_b = (-\gamma\partial+m)^a_b\,\Delta_+(x)$

$\rightarrow \Delta^D_\pm(p) = 2\pi i\,\delta(p^2+m^2)\,\Theta(\pm p^0)\,(ip\gamma+m)$ $\quad (\partial_\mu\gamma^\mu - m)\,\Delta^D_\pm(x) = 0$

Anti-commutator: $\Delta^{D\,a}{}_b(x-y) = i\{\psi^a(x),\bar{\psi}_b(y)\}$

$\rightarrow \Delta^D(x) = (\gamma\partial+m)\,\Delta(x)$, $\Delta^D(p) = 2\pi i\,\delta(p^2+m^2)\,\mathrm{sgn}(p^0)\,(ip\gamma+m)$

Propagator: $G^{D\,a}{}_b(x) = (\gamma\partial+m)^a{}_b\,G(x) \rightarrow G^D_R(x) = \Theta(t)\Delta^D(x)$

and by construction $G^D(p) = \frac{i\gamma p+m}{p^2+m^2}$

Complex Field: The Dirac Lagrangian has the $U(1)$ global symmetry: $\psi'(x) = e^{i\alpha}\psi(x)$

$\bar{\psi}'(x) = e^{-i\alpha}\bar{\psi}(x)$

$$J^\mu = \frac{\delta\psi^a}{\delta\alpha}\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^a} = +i\bar{\psi}\gamma^\mu\psi$$

$\rightarrow -J^0 = \psi^\dagger\psi$ positive definite, however

$J^\mu_c = -J^\mu$, but still conserved.

$Q = \int d\vec{x}^3 J^0 = -\int d\vec{x}^3\psi^\dagger\psi \rightarrow$ $[Q,\psi(x)] = \psi(x)$, $[Q,\bar{\psi}(x)] = -\bar{\psi}(x)$

Real Field: existence of charge conjugate solutions can be removed by imposing $\psi_c = \psi$

$u_\alpha(\vec{p}) = \gamma_c v^*_{\alpha'}(\vec{p}) \rightarrow \psi_c = \psi$ implies $a_\alpha(\vec{p}) = b_{\alpha'}(\vec{p})$

In the Weyl representation, the reality condition implies

$\psi_L = \sigma^2\psi_R^{\dagger T} =: \frac{1}{\sqrt{2}}\chi$

Majorana 2-spinor

$$\rightarrow \mathcal{L} = -\chi^\dagger i\bar{\sigma}\partial\chi + \frac{1}{2}m\chi^T\sigma^2\chi - \frac{1}{2}m\chi^\dagger\sigma^2\chi^{\dagger T}$$

Parity: interchanges $\psi_L \leftrightarrow \psi_R$

$\rightarrow \quad \chi'(t,-\vec{x}) = (CP)\chi(t,\vec{x})(CP)^{-1} = i\sigma^2\chi^{\dagger T}(t,-\vec{x})$

Two views: $C' = 1$, $P' = CP$

Massless field: Real massless Majorana field:

$\mathcal{L} = -\chi^\dagger i\bar{\sigma}\partial\chi$ has symmetry $\chi' = e^{i\alpha}\chi$

$\rightarrow J^\mu = \chi^\dagger\bar{\sigma}^\mu\chi$

At the level of 4-spinors, this corresponds to

$\psi' = \exp(-i\alpha\gamma^5)\psi$, $\bar{\psi}' = \bar{\psi}\exp(-i\alpha\gamma^5)$

$\rightarrow J^\mu = i\bar{\psi}\gamma^5\gamma^\mu\psi$

Grassmann Numbers: $\{\psi,\psi^\dagger\}\sim\hbar$ should be reflected by $\{\psi,\psi^\dagger\} = 0$ in classical theory

Grassmann numbers

- non-commutative ring: Abelian group ⊕
  Second operation ⊗: associative, distributive, with identity element
- $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ graded: $|a| = 0,1$ (even, odd)
- $|a+b| = |a| = |b|$, $|ab| = |a|+|b|$, $ab = (-1)^{|a||b|}ba$
- Basis: $a^n = \frac{1}{\sqrt{2}}(\gamma^{2n}+\gamma^{2n+1})$ $\quad \{\gamma^j,\gamma^k\} = 2\delta^{jk}$

Calculus: $(a^n)^2 = \frac{1}{2}\{a^n,a^n\} = 0 \rightarrow \sqrt{0}$ is ill-defined

all odd numbers and some even numbers (e.g. products of odd numbers) have no inverse

$$\frac{\partial}{\partial a^m}a^n = \delta^n_m \leftrightarrow \left\{\frac{\partial}{\partial a^m},a^n\right\} = \delta^n_m \rightarrow \frac{\partial}{\partial a^n} = \frac{1}{\sqrt{2}}(\gamma^{2n} - i\gamma^{2n-1})$$

Complex numbers: $a = a_r + ia_i$ $\qquad a^* = a_r - ia_i \rightarrow (ab)^* = a^*b^*$

$a^\dagger = \begin{cases} a^*, & a \text{ even} \\ -ia^*, & a \text{ odd}\end{cases} \rightarrow (ab)^\dagger = b^\dagger a^\dagger$

Therefore: $a = a^\dagger \not\rightarrow$ a real, $a^\dagger = -ia \rightarrow$ a real (for odd numbers)

Quantization: $\{F, G\} = i\int d\vec{x}^3 \left(\frac{\delta F}{\delta \psi^a(x)} \frac{\delta G}{\delta \psi^\dagger_a(x)} + \dots \right)$

$\rightarrow \{\psi^a(x), \psi^\dagger_b(y)\} = i\delta^a_b\, \delta^3(\vec{x}-\vec{y})$

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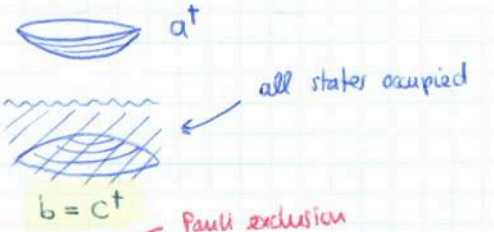

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At the level of 4-spinors, this corresponds to

$\psi' = \exp(-i\alpha\gamma^5)\psi$, $\bar{\psi}' = \bar{\psi}\exp(-i\alpha\gamma^5)$

$\rightarrow J^\mu = i\bar{\psi}\gamma^5\gamma^\mu\psi$

Grassmann Numbers: $\{\psi, \psi^\dagger\} \sim \hbar$ should be reflected by $\{\psi, \psi^\dagger\} = 0$ in classical theory

Grassmann numbers

- non-commutative ring: Abelian group $\oplus$
  Second operation $\otimes$: associative, distributive, with identity element
- $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ graded: $|a| = 0, 1$ (even, odd)
- $|a+b| = |a| = |b|$, $|ab| = |a| + |b|$, $ab = (-1)^{|a||b|}\, ba$
- Basis: $a^n = \frac{1}{\sqrt{2}}(\gamma^{2n} + \gamma^{2n+1})$ $\quad \{\gamma^j, \gamma^k\} = 2\delta^{jk}$

Calculus: $(a^n)^2 = \frac{1}{2}\{a^n, a^n\} = 0 \rightarrow \sqrt{0}$ is ill-defined

all odd numbers and some even numbers (e.g. products of odd numbers) have no inverse

$$\frac{\partial}{\partial a^m} a^n = \delta^n_m \leftrightarrow \left\{\frac{\partial}{\partial a^m}, a^n\right\} = \delta^n_m \rightarrow \frac{\partial}{\partial a^n} = \frac{1}{\sqrt{2}}(\gamma^{2n} - i\gamma^{2n-1})$$

Complex numbers: $a = a_r + ia_i$ $\qquad a^* = a_r - ia_i \rightarrow (ab)^* = a^*b^*$

$$a^\dagger = \begin{cases} a^*, & a \text{ even} \\ -ia^*, & a \text{ odd} \end{cases} \rightarrow (ab)^\dagger = b^\dagger a^\dagger$$

Therefore: $a = a^\dagger \not\longrightarrow a$ real, $a^\dagger = -ia \rightarrow a$ real (for odd numbers)

Classical Electrodynamics:

Maxwell:
$$0 = \partial_k B_k$$
$$0 = \varepsilon_{ijk}\partial_j E_k + \dot{B}_i$$
$$\rho = \partial_k E_k$$
$$J_i = \varepsilon_{ijk}\partial_j B_k - \dot{E}_i$$

Relativistic Formulation:

$$\boxed{F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_z & +B_y \\ -E_2 & +B_z & 0 & -B_x \\ -E_3 & -B_y & +B_x & 0 \end{pmatrix} = (\partial_\mu A_\nu - \partial_\nu A_\mu)} \longrightarrow \begin{matrix} \boxed{\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0} \\ \boxed{\partial_\nu F^{\nu\mu} = J^\mu} \end{matrix}$$

Gauge freedom: $\boxed{A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)}$

Lagrangian: $\boxed{\mathcal{L} = -\frac{1}{4}F^{\mu\nu}[A]F_{\mu\nu}[A]} = \frac{1}{2}\vec{E}[A]^2 - \frac{1}{2}\vec{B}[A]^2$

Energy-momentum tensor: $\boxed{T^{\mu\nu} = F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}}$

Hamiltonian Framework: Gauge Issues

$$\Pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}^\mu} = F_{0\mu}$$

$\rightarrow \boxed{\Pi_0 = 0} \rightarrow A_0$ has no conj. mom.

$\rightarrow$ e.o.m: $\boxed{\partial_k\Pi_k = 0} \rightarrow$ additional cont.

$\rightarrow$ these issues trace back to the gauge redundancy and make the Hamilton. formulation problematic

6. Free Vector Field

Coulomb gauge: $\boxed{A_0 = 0}$ to resolve $\Pi_0 = 0$

$\rightarrow$ e.o.m. $\partial_k\Pi_k = 0$ removed $\rightarrow$ the gauge-fixed system is more general than ED and

$$\vec{\partial}\dot{\vec{A}} = 0, \quad \partial_0\vec{\partial}\dot{\vec{A}} \overset{K.G.}{=} \vec{\partial}^2\vec{\partial}\vec{A} \overset{!}{=} 0$$

needs to be enforced additionally

$A_0 = 0$ does not eliminate all gauge freedom, $\alpha(\vec{x})$ remains $\rightarrow$ one may demand $\boxed{\vec{\partial}\vec{A} = 0}$

$\rightarrow \int d\vec{x}^3 \, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = H$

Lorentz gauges: $\boxed{\partial^\mu A_\mu = 0} \rightarrow \partial^2\alpha = 0$ remains

$\rightarrow$ one may demand $A_0 = 0$ to recover the Coulomb gauge

One may add $\boxed{\mathcal{L}_{gf} = -\frac{1}{2}\xi^{-1}(\partial A)^2}$ to the Lagrangian $\longrightarrow \boxed{\mathcal{L} = -\frac{1}{2}\partial^\mu A^\nu\partial_\mu A_\nu + \frac{1}{2}(1-\xi^{-1})\partial^\mu A_\mu \partial^\nu A_\nu}$

$\longrightarrow$ E.o.m: $\boxed{\partial^2 A_\mu - (1-\xi^{-1})\partial_\mu\partial^\nu A_\nu = 0} \longrightarrow$ this is again more general than ED and we have to implement $\partial A = 0$, $\partial\dot{A} \sim -\vec{\partial}^2 A_0 + \vec{\partial}\dot{\vec{A}} = 0$ by hand

Generators of Residual Symmetries:

Now $\Pi_0 = \xi^{-1}\dot{A}_0 + (1-\xi^{-1})\vec{\partial}\vec{A}$, $\vec{\Pi} = \dot{\vec{A}}$ and we impose $\{A_\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})\} = \eta_{\mu\nu}\,\delta^3(\vec{x}-\vec{y})$

$\longrightarrow \{\partial A, F\} = \xi\{-\Pi_0 + \vec{\partial}\vec{A}, F\} \neq 0$, $\{\partial\dot{A}, F\} = \xi\,\{-\vec{\partial}^2 A_0 + \vec{\partial}\vec{\Pi}, F\} \neq 0$

$\longrightarrow \partial A, \partial\dot{A}$ generate the residual gauge transf. with $\partial^2\alpha = 0$ and we cannot simply set them to 0

Feynman gauge: $\boxed{\xi = 1} \longrightarrow \boxed{\mathcal{L} = -\frac{1}{2}\partial^\mu A^\nu\partial_\mu A_\nu} \longrightarrow \boxed{\partial^2 A_\mu = 0}$

$\rightarrow$ negative sign for A_0

Particle states: $\boxed{[A_\mu(t,\vec{x}), \dot{A}_\nu(t,\vec{y})] = i\eta_{\mu\nu}\,\delta^3(x-y)} \longrightarrow$ e.o.m $\boxed{\partial^2 A_\mu = 0}$ solved by

$\boxed{A_\mu(x) = \int\frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})}\left(e^{ipx}a_\mu(\vec{p}) + e^{-ipx}a^\dagger_\mu(\vec{p})\right)}$ with $\boxed{[a_\mu(\vec{p}), a^\dagger_\nu(\vec{q})] = \eta_{\mu\nu}\,2e(\vec{p})(2\pi)^3\delta^3(\vec{p}-\vec{q})}$

$$\left.\begin{array}{l} \text{Fock space too large: 4 modes } a^\dagger_\mu \text{ instead of only 2 corresponding to } \pm h, \\ \qquad |f\rangle = \int\frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} f(\vec{p})\, a^\dagger_0(\vec{p})|0\rangle \text{ has } \langle f|f\rangle < 0 \end{array}\right\} \text{trace back to gauge freedom}$$

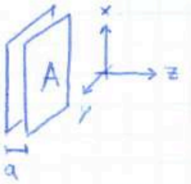
These problems can be resolved by introducing $\partial A = 0$, but we cannot do it on an operatorial level, since it has non-trivial commutation relations

physical states

Gupta-Bleuler: $\langle\Phi|\partial A|\Psi\rangle = 0 \longrightarrow \boxed{p\,a(\vec{p})|\Psi\rangle = \langle\Phi|p\,a^\dagger(\vec{p}) = 0}$

$$\text{Polarizations:} \quad \left.\begin{array}{l} \varepsilon^\mu_{(G)} = p^\mu \text{ (light-like)} \\ \varepsilon^\mu_{(L)} \text{ (light-like) with } \varepsilon_{(L)}\varepsilon_{(G)} \overset{!}{=} 1 \\ \varepsilon^\mu_{(1,2)} \text{ (orthonormal space-like)} \\ \text{with } \varepsilon_{(1,2)}\varepsilon_{(L)} = \varepsilon_{(1,2)}\varepsilon_{(G)} = 0 \end{array}\right\} \quad \boxed{\begin{array}{l} a_{(\alpha)} = \varepsilon^\mu_{(\alpha)}a_\mu \\ a^\dagger_{(\alpha)} = \varepsilon^\mu_{(\alpha)}a^\dagger_\mu \end{array}} \quad \boxed{\begin{array}{l} [a_{(L)}(\vec{p}), a^\dagger_{(G)}(\vec{q})] = [a_{(1,2)}(\vec{p}), a^\dagger_{(1,2)}(\vec{q})] = \\ [a_{(G)}(\vec{p}), a^\dagger_{(L)}(\vec{q})] = 2e(\vec{p})(2\pi)^3\delta^3(\vec{p}-\vec{q}) \end{array}}$$

Casimir Effect: tiny force between nearby conductors which exists even in absence of charges / medium

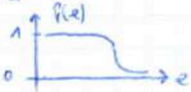


$p_z \in \frac{\pi}{a}\mathbb{Z}$ quantized

$$E = \int \frac{A\,dp_x\,dp_y}{(2\pi)^2}\left(\frac{1}{2}e(p_x,p_y,0) + 2\sum_{n=1}^{\infty}\frac{1}{2}e(p_x,p_y,\pi\frac{n}{a})\right)$$

$$= A\int_0^{\infty} \frac{p\,dp}{2\pi}\left(\frac{1}{2}p + \sum_{n=1}^{\infty}\sqrt{p^2+\pi^2\frac{n^2}{a^2}}\right)$$

high momenta will pass through the plates



$$\frac{E_{uv}}{A} = \int_0^{\infty}\frac{p\,dp}{2\pi}\int_0^{\infty} dn\,\sqrt{\cdots}\,\left(1-f(\sqrt{\cdots})\right)$$

$$\frac{E_{IR}}{A} = \frac{1}{2}F(0) + \sum_{n=1}^{\infty}F(n) = \underbrace{\int_0^{\infty} dn\,F(n)}_{E_{int}} - \overbrace{\sum_{k=1}^{\infty}(-1)^k\frac{B_{2k}}{(2k)!}F^{(2k-1)}(0)}^{E_c}$$

with $F(n) = \int_0^{\infty}\frac{p\,dp}{(2\pi)}\sqrt{\cdots}\,f(\sqrt{\cdots}) = \frac{1}{2\pi}\int_{\pi n/a}^{\infty} de\; e^2 f(e)$

$$\frac{E_0}{V} = \frac{E_{uv}+E_{int}}{Aa} = \frac{2}{4\pi^2}\int_0^{\infty} de'\int_{e'}^{\infty} de\; e^2$$

$$F(n) = F(0) - \frac{\pi^2 n^3}{6a^3}$$

$$\longrightarrow \boxed{\frac{E_c}{A}} = -\frac{B_4}{4!}F^{(3)}(0) = \boxed{-\frac{\pi^2}{720\,a^3}}$$

$$\longrightarrow \boxed{P} = \frac{F}{A} = \frac{E_c'(a)}{A} = \boxed{\frac{\pi^2}{240\,a^4}}$$

- attractive
- increases for a smaller
- quantum effect ($\hbar$, c omitted), measurable at $a \sim \mu m$
- indep. of α, e

Massive Vector Field:

Lagrangian: $\boxed{\mathcal{L} = -\frac{1}{2}\partial_\mu V_\nu \partial^\mu V^\nu + \frac{1}{2}\partial_\mu V_\nu \partial^\nu V^\mu}$ $\boxed{-\frac{1}{2}m^2 V^\mu V_\mu}$

$\rightarrow \partial^2 V_\mu - \partial_\mu \partial^\nu V_\nu - m^2 V_\mu = 0 \rightarrow$ derivative

$\rightarrow \boxed{\partial^2 V_\mu - m^2 V_\mu = 0}$ and $\boxed{\partial^\mu V_\mu = 0}$

$\boxed{\Pi_\mu = \dot{V}_\mu - \partial_\mu V_0}$

$\rightarrow$ again, Π_0 vanishes and one gets $\partial_k \Pi_k + m^2 V_0 = 0$

To solve the e.o.m, we propose for

$\Delta^V_{\mu\nu}(x-y) = [V_\mu(x), V_\nu(y)]$: $\boxed{\Delta^V_{\mu\nu}(x) = (\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu)\Delta(x}$

$\rightarrow$ We treat V_l, Π_l as fundamental fields:

$$\boxed{[V_k(\vec{x}), \Pi_l(\vec{y})] = i\delta_{kl}\,\delta^3(\vec{x}-\vec{y})}$$

$\rightarrow$ commutators involving V_0 are recovered via

$$V_0 = -m^{-2}\partial_k \Pi_k,\ \dot{V}_0 = \partial_k V_k$$

$$H = \int d\vec{x}^3\left(\frac{1}{2}\Pi_k\Pi_k + \frac{1}{2}m^{-2}\partial_k\Pi_k\partial_l\Pi_l + \frac{1}{2}\partial_k V_l \partial_k V_l - \frac{1}{2}\partial_l V_k \partial_k V_l + \frac{1}{2}m^2 V_k V_k\right)$$

$$\rightarrow \dot{V}_k = -\{H, V_k\} = \Pi_k - m^{-2}\partial_k\partial_l\Pi_l$$

$$\dot{\Pi}_k = -\{H, \Pi_k\} = \partial_l\partial_l V_k - \partial_k\partial_l V_l - m^2 V_k$$

Particle States: By construction: $\boxed{a_{(G)}(\vec{p})|\psi\rangle = 0}$

$\rightarrow \boxed{|\psi\rangle = a^{\dagger}_{(G)}\cdots a^{\dagger}_{(G)}\,a^{\dagger}_{(1,2)}\cdots a^{\dagger}_{(1,2)}|0\rangle}$ and it has $\langle\psi|\psi\rangle \geq 0$

equivalence classes

Null states: $|\psi\rangle = \underbrace{pa^{\dagger}(\vec{p})}_{a^{\dagger}_{(G)}}|\Omega\rangle$ has $\langle\psi|\psi\rangle = 0 \longrightarrow \boxed{|\psi\rangle \simeq |\psi\rangle + pa^{\dagger}(\vec{p})|\Omega\rangle}$

and we may choose $\boxed{|\psi\rangle = a^{\dagger}_{(1,2)}\cdots a^{\dagger}_{(1,2)}|0\rangle}$ as our reference states

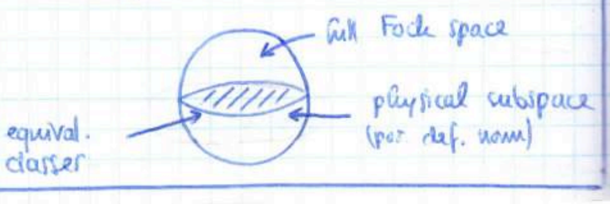
$\langle\phi|A_\mu(x)|\psi'\rangle = \langle\phi|A_\mu(x)|\psi\rangle + \langle\phi|\underbrace{[A_\mu, pa^{\dagger}(\vec{p})]}_{p_\mu e^{ipx} = -i\partial_\mu e^{ipx}}|\Omega\rangle$, hence $\boxed{A_\mu(x) \rightarrow A_\mu(x) - \frac{\langle\phi|\Omega\rangle}{\langle\phi|\psi\rangle}i\partial_\mu e^{ipx}}$

$\longrightarrow$ gauge transformation

Note: $[\partial A(x), pa^{\dagger}(\vec{p})] = -i\partial^2 e^{ipx} = 0 \rightarrow$ null states induce residual gauge transf. within the Lorentz gauge

$$\left.\begin{array}{l} [F_{\mu\nu}(x), pa^{\dagger}(\vec{p})] = 0 \\ [J[A], pa^{\dagger}(\vec{p})] = 0 \end{array}\right\}$$ the formulation is consistent!

$\uparrow\ \int dx^4\, J^\mu(x) A_\mu(x)$



7. Interactions

Scalar Interactions:

$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}\mu\phi^3 - \frac{1}{24}\lambda\phi^4 \rightarrow$ e.o.m.: $\partial^2\phi - m^2\phi - \frac{1}{2}\mu\phi^2 - \frac{1}{6}\lambda\phi^3 = 0$

↳ cannot be solved in general

↳ we get a convolution in mom. sp.

- also ϕ^5, $\phi(\partial\phi)^2$, $\phi^2(\partial\phi)^2$, $(\partial\phi)^4$, ... could be considered, but they have undesirable features regarding normalization
- only local interactions considered: $\mathcal{L}[\phi(x), \partial\phi(x), ...]$, not $\int dx^4 dy^4 f(x,y)\phi(x)\phi(y)$

Quantum Electrodynamics:

$\mathcal{L}_{QED} = \bar\psi(\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \underbrace{- iq\bar\psi\gamma^\mu\psi A_\mu}_{J^\mu} \rightarrow \begin{matrix} \partial_\mu F^{\mu\nu} = iq\bar\psi\gamma^\nu\psi \\ (\partial^\mu\gamma_\mu - m)\psi = iq\gamma^\mu A_\mu\psi \end{matrix}$

We want to preserve gauge symmetry also in the presence of interactions

$\rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x)$ and also $\psi'(x) = \exp(iq\alpha(x))\psi(x) \rightarrow$ gauge invariance made manifest

with gauge covariant derivative: $D_\mu = \partial_\mu - iqA_\mu \rightarrow D_\mu' = \exp(iq\alpha) D_\mu \exp(-iq\alpha)$

$\rightarrow \mathcal{L}_{QED} = \bar\psi(\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ and $\mathcal{L}_{SQED} = -(D^\mu\phi)^*(D_\mu\phi) - m^2|\phi|^2 - \frac{1}{4}\lambda|\phi|^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

General Interactions:

- $\bar\psi\psi\phi$ → scalar
- $i\bar\psi\gamma^5\psi\phi$ → pseudo-scalar
- $-i\bar\psi\gamma^\mu\psi A_\mu$ → vector
- $-i\bar\psi\gamma^5\gamma^\mu\psi A_\mu$ → pseudo-vector

Power Counting: $[S] = 1$, $[\mathcal{L}] = m^4$

$\rightarrow [\phi] = [A_\mu] = m$, $[\psi] = m^{\frac{3}{2}}$

All interaction terms must have $[\mathcal{L}_{int}] \leq 4$

→ otherwise: coupling constant with negative mass dim. → theory non-renormalizable

Symmetries:

- usually only few of the symmetries of the free theory survive → related to global gauge symmetries
- symmetries of classical theory not in quantum theory: anomalous

$\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi - m^2|\phi|^2 - \frac{1}{4}\lambda|\phi|^4$ invariant under

$\phi \rightarrow e^{i\alpha}\phi \rightarrow J^\mu = -i(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi)$ and

$Q = N_a - N_b$ conserved

Interaction Picture:

We want to compute $F(x_2, x_1) = \langle 0|\phi(t_2, \vec{x}_2)\phi(t_1, \vec{x}_1)|0\rangle$ with $\phi(t, \vec{x}) = \exp(iH(t-t_0))\phi(\vec{x})\exp(iH(t_0-t))$.

This can only be done approximately, we define $\phi_0(t, \vec{x}) = \exp(iH_0(t-t_0))\phi(\vec{x})\exp(iH_0(t_0-t))$

$\rightarrow \phi(t, \vec{x}) = U(t, t_0)^{-1}\phi_0(t, \vec{x})U(t, t_0)$ with $U(t, t_0) = \exp(iH_0[\phi_0](t-t_0))\exp(iH[\phi_0(t_0)](t_0-t))$

The time evolution operator satisfies the group property $U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$

↳ $X[\phi_0(t_1)]\exp(iH_0(t_1-t_0)) = \exp(iH_0(t_1-t_0))X[\phi_0(t_0)]$

Ground State: $|0_0\rangle = c_0|0\rangle + \sum_n c_n|n\rangle$

$\rightarrow \exp(-iHT)|0_0\rangle = c_0|0\rangle + \sum_n c_n\exp(-iE_nT)|n\rangle$

vanishes if T has a small negative imaginary part

$|0\rangle \simeq U(t_0, -T)|0_0\rangle$, $\langle 0| \simeq \langle 0_0|U(+T, t_0)$

$\rightarrow \langle 0|X|0\rangle = \lim_{T\rightarrow\infty(1-i\varepsilon)} \frac{\langle 0_0|U(T, t_0)XU(t_0, -T)|0_0\rangle}{\langle 0_0|U(T, -T)|0_0\rangle}$ where X is expressed in terms of $\phi(t, \vec{x}) \rightsquigarrow U(t_0, t)\phi_0(t, \vec{x})U(t, t_0)$

→ The problem was shifted to computing $U(t, t_0)$, which can be done perturbatively

Perturbation Theory:

$i\partial_t U(t, t_0) = H_{int}(t)\,U(t, t_0) = (H[\phi_0(t)] - H_0)U(t, t_0) \rightarrow U(t, t_0) = 1 - i\int_{t_0}^{t} dt'\, U(t', t_0)H_{int}(t')$

$U(t_0, t_0) = 1$

→ Dyson Series: $U(t_2, t_1) = \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}T\left[\left(\int_{t_0}^{t} dt'\, H_{int}(t')\right)^n\right] = T\left[\exp(iS_{int}(t_2, t_1))\right]$

where $S_{int}(t_2, t_1) = -\int_{t_1}^{t_2} dt'\, H_{int}(t')$

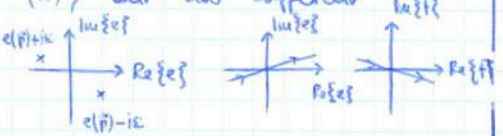
Interacting Corelators: Now that we know the form of $U(t,t_0)$, we can go a step further an see that

$$\boxed{\langle 0_{int}| T\left[X[\phi_{int}]\right] |0_{int}\rangle} = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0| T\left[X[\phi]\exp\left(iS_{int}(T,-T)\right)\right]|0\rangle}{\langle 0|T\left[\exp\left(iS_{int}(T,-T)\right)\right]|0\rangle} \simeq \boxed{\frac{\langle 0|T\left[X[\phi]\exp(iS_{int})\right]|0\rangle}{\langle 0|T\left[\exp(iS_{int})\right]|0\rangle}}$$

Time-Ordered Corelators:

Feynman propagator: $\boxed{G_F(x_1,x_2) = i\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle} \longrightarrow G_R(x) = G_F(x) - \Delta_+(-x)$,

hence G_F satisfies $-\partial^2 G_F(x) + m^2 G_F(x) = \delta^{d+1}(x)$, but has different boundary conditions: $\boxed{G_F(p) = \frac{1}{p^2+m^2-i\varepsilon}}$



Wick's theorem: $(T-N)\left[\phi(x_1)\phi(x_2)\right] = -iG_F(x_1-x_2)$

Generally: $\boxed{\langle 0|T[\phi_1,\dots,\phi_n]|0\rangle = \langle 0|N[\phi_1\dots\phi_n + \text{all possible contractions}]|0\rangle}$,

where $[\dots \underbrace{\phi_k \dots \phi_e}\dots] = -iG_F(x_k-x_e)[\dots]$

- $-i(G^D)^a{}_b$ for $\underbrace{\psi^a\bar{\psi}_b}$ for fields with spin
- (-1) for each crossing fermion line

8. Correlation Functions

Example: $\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{24}\lambda\phi^4$

→ We'd like to evaluate $\boxed{F_{1234} = \langle\phi_1\phi_2\phi_3\phi_4\rangle_{int}}$ (time-ordered)

$$F^{(0)} = \langle\phi_1\phi_2\phi_3\phi_4\rangle = \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

$$F^{(1)} = \langle\phi_1\phi_2\phi_3\phi_4\, iS_{int}[\phi]\rangle - \langle\phi_1\phi_2\phi_3\phi_4\rangle\langle iS_{int}[\phi]\rangle$$
$$= (-i\lambda)\int dy^4\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y$$
$$+ \frac{1}{2}(-i\lambda)\int dy^4\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y$$
$$+ \frac{1}{8}(-i\lambda)\int dy^4\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y$$
$$-\frac{1}{8}\dots\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

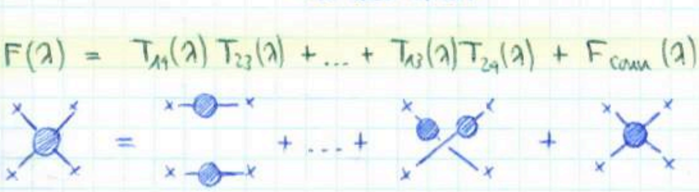
㉔ terms

⑫ terms
+ 2 permutations

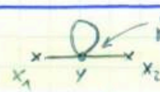
③ terms
+ 2 permutations

Disconnected Graphs: correlation functions contain disconnected products of lower-point functions
↳ these disconnected contributions represent processes that take place simultaneously without interfering with each others

$$F(\lambda) = T_{12}(\lambda)T_{23}(\lambda) + \dots + T_{13}(\lambda)T_{24}(\lambda) + F_{conn}(\lambda)$$

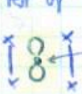


Tadpoles:



involves a $-iG_F(0)$ ⟶ divergent! We didn't consider equal times when deriving Wick's theorem (such terms do not appear) and we could eliminate the $GF(0)$ by adding $-\frac{1}{4}\lambda G_F(0)\phi^2$ to the Lagrangian.

Vacuum Bubbles: components which are coupled neither to the rest of the graph nor to external points:

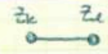
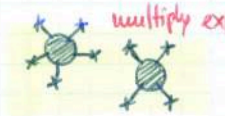


Usually infinite: $G_F(0)^2\int dy^4$, but cancel with the terms from the denomin. and can hence be discarded right away

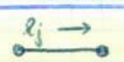
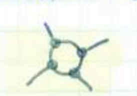
Symmetry Factors: when the symmetry factors of the Lagrangian are set up properly, the factor in front of the corelator term is $\boxed{\frac{1}{n}}$, where n is the number of internal symmetries.

→ $n = 1$ → $n = 2$ → $n = 8$

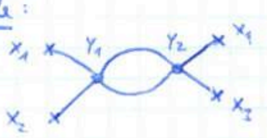
Feynman Rules in Position Space: Permissible graphs for $\langle \phi(x_1) \dots \phi(x_n) \rangle_{int}$:

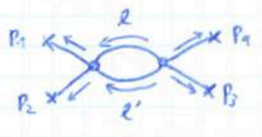
- have edges  $-i\,G_F(z_k - z_l)$
- have 1-valent (external) vertices 
- have 4-valent (internal) vertices  $-i\lambda \int dy_j^4$
- can have tadpoles 
- can have multiple lines connecting two vertices 
- can have several connection components 
- must not have vacuum bubbles : ⊗
- terms are multiplied by symmetry factor

Feynman Rules in Momentum Space: Permissible graphs for $\int dx_1^4 \dots dx_n^4 \, e^{-ix_1p_1 - \dots - ix_np_n} \langle \phi_1 \dots \phi_n \rangle_{int}$:

- have edges  $\frac{-i}{\ell_j^2 + m^2 - i\varepsilon}$
- have 1-valent (external) vertices 
- have 4-valent (internal) vertices  $-i\lambda$
- the momentum is conserved at each vertex, there needs to be a factor enforcing overall momentum conservation: $(2\pi)^4 \delta^4(p_1 + \dots + p_n)$
- have loops  $\int \frac{d\ell_j^4}{(2\pi)^4}$
- terms are multiplied by symmetry factor

Example:





$$F = \frac{1}{2}(-i\lambda)^2 \int dy_1^4 dy_2^4 \; (-i)^6 G_F(x_1 - y_1) G_F(x_2 - y_1) \cdots G_F(x_3 - y_2)$$

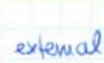
$$F = \frac{1}{2}(-i\lambda)^2 (-i)^6 (2\pi)^4 \delta^4(p_1 + .. + p_4) \prod_{j=1}^{4} \frac{1}{p_j^2 + m^2 - i\varepsilon}$$
$$\int \frac{d\ell^4}{(2\pi)^4} \frac{1}{\ell^2 + m^2 - i\varepsilon} \frac{1}{(p_1 + p_2 - \ell)^2 + m^2 - i\varepsilon}$$

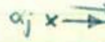
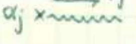
Feynman Rules for QED:

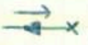
$$\mathcal{L} = \bar{\Psi}(\gamma^\mu \partial_\mu - m)\Psi - \frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu - iq \bar{\Psi} \gamma^\mu \Psi A_\mu$$

Scattering amplitude proportional to $\langle q_1, \dots, q_n | S | p_1, \dots, p_m \rangle$

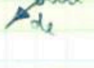
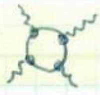
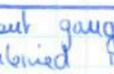
Permissible Feynman graphs for QED interactions:

- have edges  $\frac{-i(\overleftarrow{-i\ell_j\gamma} + m)^{c_k}{}_{d_l}}{\ell_j^2 + m^2 - i\varepsilon}$ for momentum in direction of fermion line  $\frac{-i\eta_{\nu_k\nu_l}}{\ell_j^2 - i\varepsilon}$
- have 1-valent external vertices

 $\sqrt{Z_\Psi}\, \bar{v}_{\alpha_j}(\vec{p}_j)$  $\sqrt{Z_\Psi}\, v_{\alpha_k}(\vec{q}_k)$  $\sqrt{Z_A}\, \varepsilon_{\alpha_j}(\vec{p}_j)$

 $\sqrt{Z_\Psi}\, u_{\alpha_j}(\vec{p}_j)$  $\sqrt{Z_\Psi}\, \bar{u}_{\alpha_k}(\vec{q}_k)$  $\sqrt{Z_A}\, \varepsilon^*_{\alpha_k}(\vec{q}_k)$

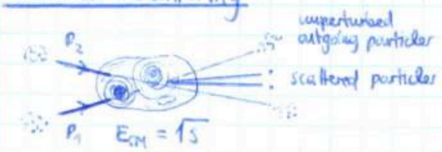
} derived in chapter 10

- have internal vertices  $-q(\gamma^{\nu_j})^{c_k}{}_{d_l}$
- can have fermion loops  $(-1)\int \frac{d\ell_j^4}{(2\pi)^4}$
- cutting the graph at any internal line must not split off a graph with two external lines 
- if two external particles are connected directly:  $2e(\vec{p}_j)(2\pi)^3 \delta^3(\vec{p}_j - \vec{q}_k)$

} derived in chapter 10

Gauge Invariance: Feynman gauge is very convenient, but gauge-invariant results are only to be expected when the fields are combined in a gauge-invariant manner, e.g. $F_{\mu\nu}(x)$ or $\int dx^4 \, J^\mu(x) A_\mu(x)$

General Scattering :



$E_{CM} = \sqrt{s}$

$$N = \frac{n_{exp} n_1 n_2 \sigma}{A}$$

with $\lim_{t_{out,in} \to \pm\infty} \langle f | \exp(-iH(t_{out}-t_{in})) | i \rangle = (2\pi)^4 \delta^4(P_{in}-P_{out})\, iM$

$$\sigma \sim |\langle f | \exp(-iH(t_{out}-t_{in})) | i \rangle|^2 \quad \langle f| \sim \langle q_1, \dots, q_n| \quad |i\rangle \sim |p_1, p_2\rangle$$

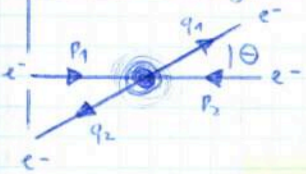
$$d\sigma = \frac{(2\pi)^4 \delta^4(P_{in}-P_{out})}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \prod_{k=1}^{n} \frac{d\vec{q}_k^{\,3}}{(2\pi)^3 2e(\vec{q}_k)} |M|^2$$

$$\frac{d^2\sigma}{d\Omega} = \frac{1}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \frac{|\vec{q}_1|}{16\pi^2\sqrt{s}} |M|^2 = \frac{|M|^2}{64\pi^2 s}$$

2-body scattering

all masses equal

Møller Scattering :



$|i\rangle = a_\alpha^+(\vec{p}_1)\, a_\beta^+(\vec{p}_2)\, |0\rangle \qquad p_{1,2} = (e, 0, 0, \pm p)$

$\langle f| = \langle 0|\, a_\delta(\vec{q}_2)\, a_\gamma(\vec{q}_1) \qquad q_{1,2} = (e, \pm p\sin\Theta, 0, \pm p\cos\Theta)$

$\Theta \to -\Theta$ symmetry (φ)

$\Theta \to \pi-\Theta$ symmetry (e^-)

in the interaction picture, $F = \langle f | U_{int}(t_{out}, t_{in}) | i \rangle = (2\pi)^4 \delta^4(P_{in}-P_{out})\, iM$

$$F^{(0)} = \langle f | i \rangle = \begin{array}{l} 2e(\vec{p}_1) 2e(\vec{p}_2) (2\pi)^6 \delta^3(\vec{p}_1-\vec{q}_1)\delta^3(\vec{p}_2-\vec{q}_2) \\ -2e(\vec{p}_1) 2e(\vec{p}_2) (2\pi)^6 \delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_1) \end{array}$$

$F^{(1)} = i\langle f | S_{int} | i \rangle =$

$q \int dx^4 \langle f | A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) | i \rangle = 0$

$F^{(2)} = i^2 \frac{1}{2} \langle f | S_{int}^2 | i \rangle =$

$\frac{1}{2} q^2 \int dx^4 dy^4 \langle f | T\left[A_\mu(x) \bar{\psi}(x)\gamma^\mu\psi(x) A_\nu(y) \bar{\psi}(y)\gamma^\nu\psi(y)\right] | i \rangle$

$F^{(2)}_{conn} = -\frac{1}{2} q^2 i \int dx^4 dy^4\, G^F_{\mu\nu}(x-y) \langle f | N\left[\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(y)\gamma^\nu\psi(y)\right] | i \rangle$

$$\{\bar{\psi}(x), a_\alpha^+(\vec{p})\} = e^{ipx}\bar{v}_\alpha(\vec{p}) \qquad [A_\mu(x), \tilde{a}_\sigma^+(\vec{p})] = \varepsilon_{\sigma,\mu} e^{ipx}$$
$$\{a_\alpha(\vec{q}), \psi(x)\} = e^{-iqx} v_\alpha(\vec{q}) \qquad [\tilde{a}_\sigma(\vec{q}), A_\mu(x)] = \varepsilon^*_{\sigma,\mu} e^{-iqx}$$

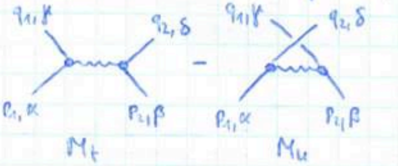
$= -iq^2 \int dx^4 dy^4\, G^F_{\mu\nu}(x-y)\, e^{ip_1x + \dots - iq_2y}\, v_\alpha(\vec{p}_1)\gamma^\mu v_\gamma(\vec{q}_1) v_\beta(\vec{p}_2)\gamma^\nu v_\delta(\vec{q}_2)$

$+ \dots$ (same term $q_1 \leftrightarrow q_2$ and $\gamma \leftrightarrow \delta$) ...

$$\int dx^4 dy^4\, G^F_{\mu\nu}(x-y)\, e^{ip_1x + \dots - iq_2y} = (2\pi)^4 \delta^4(P_{out}-P_{in})\, G^F_{\mu\nu}(q_1-p_1)$$

$$M_t = -\frac{q^2 \eta_{\mu\nu}}{(p_1-q_1)^2} \bar{v}_\alpha(\vec{p}_1)\gamma^\mu v_\gamma(\vec{q}_1)\bar{v}_\beta(\vec{p}_2)\gamma^\nu v_\delta(\vec{q}_2)$$

$M_u = -$ .... ($q_1 \leftrightarrow q_2$ and $\gamma \leftrightarrow \delta$)



using $\sum_\alpha v_\alpha \bar{v}_\alpha = ip\gamma - m$

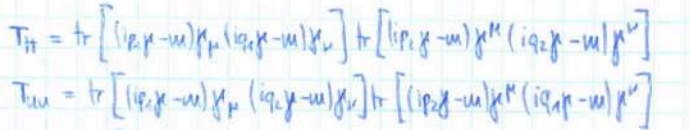
$$|M|^2 = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} M_{\alpha\beta\gamma\delta} M^*_{\alpha\beta\gamma\delta} =$$

$$\frac{q^4\, T_{tt}}{4(p_1-q_1)^4} + \frac{q^4\, T_{uu}}{4(p_1-q_2)^4} - \frac{q^4\, T_{tu}}{2(p_1-q_1)^2(p_1-q_2)^2}$$

$T_{tt} = tr\left[(ip_1\gamma - m)\gamma_\mu(iq_1\gamma - m)\gamma_\nu\right] tr\left[(ip_2\gamma - m)\gamma^\mu(iq_2\gamma - m)\gamma^\nu\right]$

$T_{uu} = tr\left[(ip_1\gamma - m)\gamma_\mu(iq_2\gamma - m)\gamma_\nu\right] tr\left[(ip_2\gamma - m)\gamma^\mu(iq_1\gamma - m)\gamma^\nu\right]$

$T_{tu} = tr\left[(ip_1\gamma - m)\gamma_\mu(iq_1\gamma - m)\gamma^\nu(ip_2\gamma - m)\gamma^\mu(iq_2\gamma - m)\gamma_\nu\right]$



$$|M|^2 = q^4\left(\frac{u-s}{t} + \frac{t-s}{u}\right)^2 + \frac{16q^4m^2(5m^2-2s)}{tu}$$

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{64\pi^2 e^2}\left(\frac{(4p^2+3m^2)^2}{p^4\sin^4\Theta} - \frac{8p^4+12m^2p^2+3m^4}{p^4\sin^2\Theta} + 1\right)$$

↳ diverges for small p, $\Theta \approx 0, \pi$ (σ diverges for $\cos\Theta = 1$)

9. Particle Scattering

Trace Identities :

$tr(1) = 4 \qquad tr(\gamma^\mu) = 0$

$tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \qquad tr(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

Enveloping Identities :

$\gamma^\mu\gamma_\mu = 4 \qquad \gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$

$\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu = 4\eta^{\nu\rho} \qquad \gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

Mandelstam Invariants :

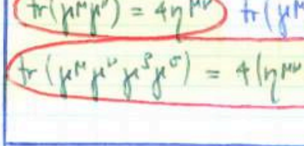
$s+t+u = 4m^2$

$s = -(p_1+p_2)^2 = -(q_1+q_2)^2$

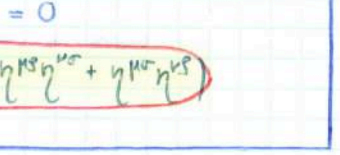
$t = -(p_1-q_1)^2 = -(p_2-q_2)^2$

$u = -(p_1-q_2)^2 = -(p_2-q_1)^2$

Crossing Symmetry → Bhabha Scattering



Since



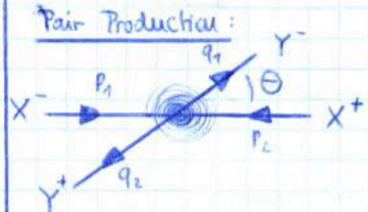
$\vec{q}_2 \longleftrightarrow -\vec{p}_2$

and $\sum_\alpha v_\alpha(\vec{p}_2)\bar{v}_\alpha(\vec{p}_2) \leftrightarrow -\sum_\alpha u_\alpha(\vec{q}_2)\bar{u}_\alpha(\vec{q}_2)$

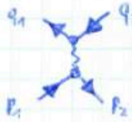
$$|M|^2 = q^4\left(\frac{s-u}{t} + \frac{t-u}{s}\right)^2 + \frac{16q^4m^2(5m^2-2u)}{st}$$

outgoing fermion line can be related to an incoming anti-fermion line via $P_{out} \to -P_{in}$ and vice versa

Pair Production:



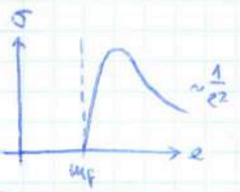
Incoming particles: $m_i, \pm q_i$
Outgoing particles: $m_f, \pm q_f$



Fermions → fermions :

$$T = \mathrm{tr}\left[(ip_1\gamma - m)\gamma_\mu (ip_2\gamma + m)\gamma_\nu\right] \mathrm{tr}\left[(iq_1\gamma - m)\gamma^\mu (iq_2\gamma + m)\gamma^\nu\right]$$

$$\rightarrow |M|^2 = \frac{q_i^2 q_f^2 T}{4s^2} = q_i^2 q_f^2 \left(\frac{e^2-m_i^2}{e^2}\frac{e^2-m_f^2}{e^2}\cos^2\Theta + \frac{m_i^2+m_f^2}{e^2} + 1\right)$$



$$\frac{d^2\sigma}{d\Omega} = \sqrt{\frac{e^2-m_f^2}{e^2-m_i^2}}\ \frac{|M|^2}{256\pi^2 e^2} \longrightarrow \boxed{\sigma = \frac{q_i^2 q_f^2}{48\pi e^2}\sqrt{\frac{e^2-m_f^2}{e^2-m_i^2}}\ \frac{e^2+\frac{1}{2}m_i^2}{e^2}\ \frac{e^2+\frac{1}{2}m_f^2}{e^2}}$$

Scalars → scalars : $\sigma = \frac{q_i^2 q_f^2}{192\pi e^2}\sqrt{\frac{e^2-m_f^2}{e^2-m_i^2}}\ \frac{e^2-m_i^2}{e^2}\ \frac{e^2-m_f^2}{e^2}$

Spinors → scalars : $\sigma = \text{.........}\ \frac{e^2+\frac{1}{2}m_i^2}{e^2}\ \text{.....}$

Scalars → spinors : $\sigma = \frac{q_i^2 q_f^2}{48\pi e^2}\ \text{..........}\ \frac{e^2+\frac{1}{2}m_f^2}{e^2}$

makes sense: photon is a spin-1 particle!

spin-0 couples with $\boxed{e^2-m^2}$ to γ → $\boxed{e^2-m^2}$ for scalars

spin-1 couples with $\boxed{e^2+m^2}$ to γ → $(e^2-m^2)+3(e^2+m^2) = \boxed{4(e^2+\frac{1}{2}m^2)}$ for spinors

Loop Contributions:

$\int \frac{dk^4}{k^4} \rightarrow$ diverges

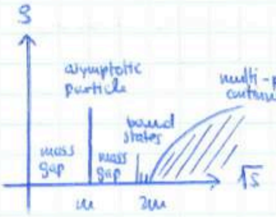
Feynman propagator to be evaluated right on-shell → diverges

Källen-Lehmann: $\Delta_+(x-y) = i\langle 0|\phi_{int}(x)\phi_{int}(y)|0\rangle$

correlator for free field with $-p^2 = s$

$\rightarrow$ Poincaré symmetry : $\boxed{\Delta_+(x)} = i\int\frac{dp^4}{(2\pi)^4}e^{ipx}\Theta(p^0)\rho(-p^2) = i\int_0^\infty\frac{ds}{2\pi}\rho(s)\int\frac{dp^4}{(2\pi)^4}e^{ipx}\Theta(p^0)2\pi\delta(p^2+s^2) = \boxed{\int_0^\infty\frac{ds}{2\pi}\rho(s)\Delta_+(s;x)}$

Spectral function:



field strength / wavefunction renormalization

$$\boxed{\rho(s) = 2\pi Z\,\delta(s-m^2) + \text{bound states} + \text{continuum}}$$

Asymptotic particles : $\boxed{\phi(x) = \underbrace{\sqrt{Z}\,\phi_{as}(x)}_{\sim a^+_{as}+a_{as}} + \underbrace{\text{bound states}}_{\sim a^{+n}_{as}+a^n_{as}} + \underbrace{\text{continuum} + \dots}_{a^{+m}_{as}a^n_{as}}}$

$$\boxed{\phi_{as}(x) = \int\frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})}\left(e^{ipx}a_{as}(\vec{p}) + \text{h.c.}\right)}$$

$\rightarrow H_{as} = \int\frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})}e(\vec{p})\,a^+_{as}(\vec{p})a_{as}(\vec{p})$

- $H_{as}|0\rangle = 0$
- $H_{as}a^+_s(\vec{p})|0\rangle = e(\vec{p})a^+_s(\vec{p})|0\rangle$

Commutator : $\Delta(x-y) = i\langle 0|[\phi_{ins}(x),\phi_{ins}(y)]|0\rangle$

$\rightarrow \Delta(x) = \int_0^\infty\frac{ds}{2\pi}\rho(s)\Delta(s;x)$

$\rightarrow \mathring{\Delta}(0,\vec{x}) = \int_0^\infty\frac{ds}{2\pi}\rho(s)\mathring{\Delta}(s;0,\vec{x}) = -i\delta^3(\vec{x})\int_0^\infty\frac{ds}{2\pi}\rho(s) \overset{!}{=} -i\delta^3(\vec{x})$

$\rightarrow \boxed{\int_0^\infty\frac{ds}{2\pi}\rho(s) \overset{!}{=} 1}$ for the field to be canonically normalized

10. Scattering Matrix

S-Matrix : $|f\rangle = \exp(-iH(t_{out}-t_{in}))|i\rangle$

Schröd. picture :

states still evolving

Interaction picture : $|out\rangle = \exp(iH_{as}t_{out})|f\rangle$

$|in\rangle = \exp(iH_{as}t_{in})|i\rangle$

$$|out\rangle = \underbrace{\exp(iH_{as}t_{out})\exp(-iH(t_{out}-t_{in}))\exp(-iH_{as}t_{in})}_{\boxed{U_{as}(t_{out},t_{in})}}|in\rangle$$

Interaction picture :

$$\boxed{S = \lim_{t_{in,out}\to\mp\infty}U_{as}(t_{out},t_{in}) = U_{as}(+\infty,-\infty)}$$

$|out\rangle = S|in\rangle$ $\langle out|S-1|in\rangle = \begin{matrix}(2\pi)^4\delta^4(P_{in}-P_{out})\\ iM(p_1,\dots,p_m;q_1,\dots,q_n)\end{matrix}$

$S|0\rangle = |0\rangle$, $S|\vec{p}\rangle = |\vec{p}\rangle$

$S^+ = S^{-1}$ unitary, $U(w,a)SU(w,a)^{-1} = S$

LSZ - Reduction: How does S relate to time-ordered correlator values?

To recover a^+_{as}, a_{as} from $\phi(x) = \sqrt{Z}\int\frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})}\left(e^{ipx}a_{as}(\vec{p}) + e^{-ipx}a^+_{as}(\vec{p})\right) + \dots$

We use $F(e) = \int_{t_1}^{t_2}dt\,e^{-iet}f(t) = \frac{ic}{e-w}\left(e^{-i(e-w)t_2} - e^{-i(e-w)t_1}\right)$

$f(t) = ce^{iwt}$

$\rightarrow F(e) = \int_{-\infty}^{t_2}dt\,e^{-iet}f(t) = \boxed{\frac{ic}{e-w}} + \text{finite}$

we project out the pole

$\rightarrow \int_{-\infty}^{t_2}dt\int d\vec{x}\,e^{ipx}\phi(x) = \frac{i\sqrt{Z}}{p^2+m^2}\left(\Theta(-e)a_{in}(-\vec{p}) - \Theta(e)a^+_{in}(\vec{p})\right) + \dots$

with $a_{in}(\vec{p}) = U_{as}(0,-\infty)\,a_{as}(\vec{p})\,U_{as}(-\infty,0)$

$\int_{t_1}^{\infty}dt\int d\vec{x}^3\,e^{ipx}\phi(x) = -\frac{i\sqrt{Z}}{p^2+m^2}\left(\Theta(-e)a_{out}(-\vec{p}) - \Theta(e)a^+_{out}(\vec{p})\right) + \dots$

with $a_{out}(\vec{p}) = U_{as}(0,+\infty)\,a_{as}(\vec{p})\,U_{as}(+\infty,0)$

outgoing momenta

Hence: $F_{m,n}(p,q) = \int\prod_{k=1}^{m}\left(dx^4_k\,e^{ip_kx_k}\right)\prod_{l=1}^{n}\left(dy^4_l\,e^{-iq_ly_l}\right)\langle 0|T[\phi(x_1)\dots\phi(x_m)\phi(y_1)\cdots\phi(y_n)]|0\rangle$

$X = \int dx^4\,e^{ipx}\,T[\phi(x)Y] \simeq \frac{-i\sqrt{Z}}{p^2+m^2}\left(T[Y]a^+_{in}(\vec{p}) - a^+_{out}(\vec{p})T[Y]\right)$

$X = \int dy^4\,e^{-iqy}\,U_{as}(+\infty,0)T[\phi(y)Y]U_{as}(0,-\infty) \simeq \frac{-i\sqrt{Z}}{q^2+m^2}\left[U_{as}(+\infty,0)T[Y]U_{as}(0,-\infty), a_{as}(\vec{q})\right]$

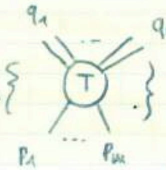
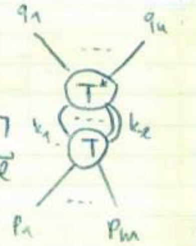
$$\rightarrow \boxed{F_{m,n} \simeq \prod_{k=1}^{m}\left(\frac{-i\sqrt{Z}}{p_k^2+m^2}\right)\prod_{k=1}^{n}\left(\frac{-i\sqrt{Z}}{q_k^2+m^2}\right)\langle 0|\left[a_{as}(\vec{q}_1),\dots\left[a_{as}(\vec{q}_n),S\right]\dots\right]a^+_{as}(\vec{p}_1)\cdots a^+_{as}(\vec{p}_m)|0\rangle}$$

$\rightarrow$ S-matrix is related to time-ordered expectation values

Unitarity : $S^\dagger = S^{-1}$ unitary

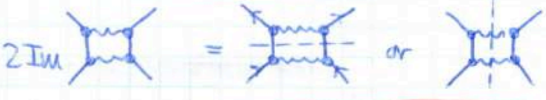
Optical theorem: $\boxed{S = 1 + iT}$

$\rightarrow \boxed{2\mathrm{Im}\{T\} = T^\dagger T = TT^\dagger}$

2Im{  } $= \underbrace{\sum_{l=2}^{\infty}\prod_{j=1}^{l}\int\frac{d\vec{k}_j^{\,3}}{(2\pi)^3 2e(\vec{k}_j)}}_{}\sum_{pol}$ 

note that T represents the entirety of all diagrams here

connecting lines have to be on-shell with directed flow of energy from T to $T^\dagger$



Tree level: $\boxed{\frac{1}{p^2+m^2-i\varepsilon} = \frac{1}{p^2+m^2} + i\pi\,\delta(p^2+m^2)} \longrightarrow$ Im{T} has $\frac{1}{p^2+m^2-i\varepsilon} - \frac{1}{p^2+m^2+i\varepsilon} = 2\pi i\,\delta(p^2+m^2)$

S-Matrix Reconstruction:

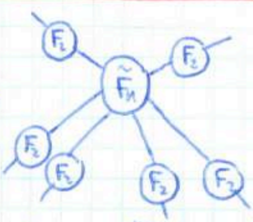
$F_{1,1} \simeq \frac{-i\sqrt{Z}}{p^2+m^2}\,\frac{-i\sqrt{Z}}{q^2+m^2}\langle 0|a_{as}(\vec{q})(S-1)a^\dagger_{as}(\vec{p})|0\rangle$ vanishes ⟶ there is no double pole!

Instead: $F_2(x-y) = \langle 0|T[\phi_{int}(x)\phi_{int}(y)]|0\rangle = -i\int_0^\infty\frac{ds}{2\pi}\rho(s)G_F(s;x-y)$

$$\rightarrow \boxed{F_2(p)} = -i\int_0^\infty\frac{ds}{2\pi}\frac{\rho(s)}{p^2+s-i\varepsilon} = \underbrace{\frac{-iZ}{p^2+m^2-i\varepsilon}}_{\text{single pole} \rightarrow \text{indicates stable particle}} + \dots$$

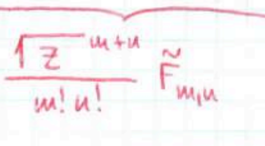
The LSZ formula can be inverted to find

$$\boxed{S = 1 + \sum_{n,m=2}^{\infty}\int\prod_{k=1}^{n}\frac{d\vec{q}_k^{\,3}}{(2\pi)^3}\frac{a^\dagger_{as}(\vec{q}_k)}{2e(\vec{q}_k)}\prod_{k=1}^{m}\frac{d\vec{p}_k^{\,3}}{(2\pi)^3}\frac{a_{as}(\vec{p}_k)}{2e(\vec{p}_k)}\underbrace{\left(\frac{F_{m,n}(p,q)}{m!\,n!}\prod_{k=1}^{m}\overset{\frac{\sqrt{Z}}{F_2(p_k)}}{\frac{p_k^2+m^2}{-i\sqrt{Z}}}\prod_{k=1}^{n}\overset{\frac{\sqrt{Z}}{F_2(q_k)}}{\frac{q_k^2+m^2}{-i\sqrt{Z}}}\right)}_{\frac{\sqrt{Z}^{\,m+n}}{m!\,n!}\tilde{F}_{m,n}}}$$

If we have  = 

$F_{m,n}(p,q)$

$$\underbrace{\tilde{F}_{m,n}(p,q)}_{\text{amputated diagram}}\underbrace{\prod_{k=1}^{m}F_2(p_k)\prod_{k=1}^{n}F_2(q_k)}_{\text{external legs}}$$

→ solves the problem of 

Self-Energy: We use $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{6}\kappa\mu\phi^3 - \frac{1}{24}\kappa^2\lambda\phi^4$ → consider $F_2(p,q) = -i(2\pi)^4\delta^4(p+q)M_2(p)$

M_2 has the contributions

$$M_2^{(0)} = \frac{1}{p^2+\mu^2-i\varepsilon} \qquad M_2^{(2)} = \frac{i(-i\mu\kappa)^2(-i)^4}{(p^2+\mu^2-i\varepsilon)^2} i I(-p^2) \qquad \text{ignored} \qquad M_2^{(4)} = \frac{\mu^4\kappa^4 I(-p^2)^2}{(p^2+\mu^2-i\varepsilon)^3}$$

appears to have a double pole

where $$I(-p^2) = \frac{1}{2}\int\frac{-i\,d\ell^4}{(2\pi)^4}\frac{1}{\ell^2+\mu^2-i\varepsilon}\frac{1}{(p-\ell)^2+\mu^2-i\varepsilon}$$

Summing all higher-order diagrams with loops, we get $$\frac{1}{p^2+\mu^2-i\varepsilon}\sum_{k=0}^{\infty}\left(\frac{\mu^2\kappa^2 I(-p^2)}{p^2+\mu^2-i\varepsilon}\right)^k = \frac{1}{p^2+\mu^2-\mu^2\kappa^2 I(-p^2)-i\varepsilon}$$

→ to leading order: $m^2 = \mu^2 - \mu^2\kappa^2 I(\mu^2) + \ldots$ → residue: $p^2+\mu^2-\mu^2\kappa^2 I(-p^2) = (p^2+m^2)(1+\mu^2\kappa^2 I'(\mu^2)) + \ldots$

→ $Z = 1 - \mu^2\kappa^2 I'(\mu^2) + \ldots$

Spectral Function: $M_2(p) = \int_0^\infty \frac{ds}{2\pi}\frac{\rho(s)}{p^2+s-i\varepsilon}$ → using that $\frac{1}{x-i\epsilon} = \frac{1}{x} + i\pi\delta(x)$ → $\rho(-p^2) = 2\,\mathrm{Im}\{M_2(p)\}$

→ expanding $M_2(p)$ in $M_2^{(0)}(p) + M_2^{(2)}(p)$ and using $\frac{1}{(x-i\varepsilon)^2} = \frac{1}{x^2} - i\pi\delta'(x)$: $$\rho(s) = 2\pi Z\,\delta(s-m^2) + \frac{2\mu^2\kappa^2\,\mathrm{Im}\{I(s)\}}{(s-\mu^2)^2}$$

as expected.

asymptotic particle

multi-part. continuum

11. Loop Corrections

Regularisation Schemes:

- Cutoff: $p < \Lambda_{cut}$ or $p > \Lambda_{cut}$
  not easy to formulate consistently for all orders
- Pauli-Villars: $\frac{1}{p^2+m^2-i\varepsilon} \longrightarrow \frac{1}{p^2+m^2-i\varepsilon} - \frac{1}{p^2+M^2-i\varepsilon}$
- Point-Splitting: in position space, UV divergences are related to putting several fields at the same position in spacetime → separate field's insertion points by tiny amount in the action
- Lattice: use finite lattice
- Dimensional Regularization: take D (number of dim.) as real/complex number → observables become function in D that have divergences of the form $\frac{1}{D-4}$
- Finite Observables: consider physical observables only: e.g. in our example, all observables can be deduced from $I'(s)$, which is finite

Renormalization:

- all physical relevant information and all observables for a QFT model are encoded into its quantum correlation function
- $\mathcal{L}/\mathcal{H}$ are devices to derive suitable correlation functions, they are not physical
- correlation function depend on bare parameters → tune μ, κ to get the right physical values:

$$\mu = \mu(m, \Lambda_{cut}) = m + \frac{1}{2}m\kappa^2 I(m^2) + \ldots$$

$$I(m^2) = \frac{2-\pi/\sqrt{3}}{32\pi^2} - \frac{1}{32\pi^2}\log\left(\frac{m^2}{\Lambda_{cut}^2}\right)$$

Loop Integral: $$\frac{1}{AB} = \int_0^1\frac{dz}{(zA+\bar{z}B)^2} \quad \text{Feynman}$$

$$\frac{1}{A} = \int_0^\infty dz\, e^{-Az} \quad \text{Schwinger}$$

$$I(-p^2) = \int_0^1 dz\int\frac{-i\,d\ell^4}{32\pi^4}\frac{1}{(z\ell^2+\bar{z}(\ell-p)^2+\mu^2-i\varepsilon)} \overset{\ell\to\ell+\bar{z}p}{=}$$

$$\int_0^1 dz\int\frac{-i\,d\ell^4}{32\pi^4}\frac{1}{(\ell^2+z\bar{z}p^2+\mu^2-i\varepsilon)^2} \overset{\text{complex analysis}}{=}$$

$$\int_0^1 dz\int\frac{d\vec{\ell}^3}{64\pi^3}\frac{1}{(\vec{\ell}^2+z\bar{z}p^2+\mu^2-i\varepsilon)^{\frac{3}{2}}} = \ldots =$$

$$-\frac{1}{32\pi^2}\int_0^1 dz\,\log\left(\frac{z\bar{z}p^2+\mu^2-i\varepsilon}{\Lambda_{cut}^2}\right)$$

integrals over space also decrease exponent by $-\frac{1}{2}$, momentum cutoff $|\vec{\ell}| < \Lambda_{cut}$ in last integral

Wick Rotation: $\ell^0 = i\ell_E^4$

the fourth component of an Euclidean vector

→ Residue theorem: $\int d\ell^4 F(\ell^0,\vec{\ell}) = \int i\,d\ell_E^4\, F(i\ell_E^4,\vec{\ell})$

Here: $$I(-p^2) = \int_0^1 dz\int\frac{d\ell_E^4}{32\pi^4}\frac{1}{(\ell_E^2+z\bar{z}p^2+\mu^2-i\varepsilon)^2} = \int_0^1 dz\int_0^{\infty}\frac{d\ell_E}{16\pi^2}\frac{\ell_E^3}{(\ell_E^2+z\bar{z}p^2+\mu^2-i\varepsilon)^2}$$

volume of three sphere is $2\pi^2|\ell_E|^3$

∞ – cutoff

Final Integral: $$I(-p^2) = -\frac{1}{16\pi^2}\sqrt{\frac{p^2+4\mu^2-i\varepsilon}{-p^2}}\,\mathrm{arct}\left(\sqrt{\frac{-p^2}{p^2+4\mu^2-i\varepsilon}}\right) - \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{\Lambda_{cut}^2 e^2}\right)$$

$$\longrightarrow \mathrm{Im}\{I(-p^2)\} = \begin{cases} 0, & -p^2 < 4\mu^2 \\ \frac{1}{32\pi}\sqrt{\frac{-p^2-4\mu^2}{-p^2}}, & -p^2 > 4\mu^2 \end{cases}$$

multi-particle creation channel opens

Spectral Function: $$\rho(s) = 2\pi Z\,\delta(s-m^2) + \frac{\mu^2\kappa^2\Theta(s-4\mu^2)}{16\pi(s-\mu^2)^2}\sqrt{\frac{s-4\mu^2}{s}}$$

$$\int\frac{ds}{2\pi}\rho(s) = Z + \kappa^2\frac{2\sqrt{3}\pi-9}{288\pi^2} + \ldots = 1$$

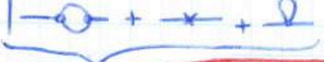
Counterterms: In our example, the overall dependence of $I(-p^2)$ on p^2 is an arctan, however, the divergent or cutoff-dependence is much simpler: $\frac{d}{d\Lambda_{cut}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{cut}}$

Divergences typically couple to polynomials of the momenta only → translate to localised distributions such as $\delta^4(x)$ under a Fourier transformation

 diverges as r^{-4} for small r → divergence localised in spacetime and can be absorbed by a  local term in the Lagrangian.

Counterterm for Self-Energy: $\mathcal{L}_{as} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$

$$\mathcal{L}_{int} = \underbrace{-\frac{1}{2}\mu' m^2\phi^2}_{\text{counterterm}} - \frac{1}{6}\kappa m\phi^3 - \frac{1}{24}\lambda\kappa^2\phi^4$$

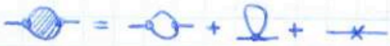


$$M_2(p) = \frac{1}{p^2 + m^2 + \mu' m^2 - m^2\kappa^2 I(-p^2) + i\varepsilon} \rightarrow m^2 \overset{\text{pole still at the same position}}{=} m^2 + \mu' m^2 - m^2\kappa^2 I(m^2)$$

for $\mu' = \kappa^2 I(m^2)$

$$\rightarrow M_2(p) = \frac{1}{p^2 + m^2 - m^2\kappa^2 I_{sub}(-p^2) - i\varepsilon} + \dots, \text{ where}$$

$$I_{sub}(-p^2) = I(-p^2) - I(m^2)$$



Similarly: counterterm $\mathcal{L}_{int} = -\frac{1}{2}\xi(\partial\phi)^2$ to change the overall normalization of the field $\phi(x)$ to 1
(we can drop the factor Z)

Vertex renormalization:

 are all divergent

$$\rightarrow \mathcal{L}_{as} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_{int} = -\frac{1}{6}\kappa m\phi^3 - \frac{1}{24}\lambda\kappa^2\phi^4$$

$$\mathcal{L}_{ct} = \underbrace{-\frac{1}{2}\mu' m^2\phi^2}\underbrace{-\frac{1}{6}\kappa'\kappa m\phi^3}\underbrace{-\frac{1}{24}\lambda'\kappa^2\phi^4}$$

→ to make all observables finite:

$$\mu' = \kappa^2 I(m^2), \quad \kappa' = 3\kappa^2\lambda I(m^2), \quad \lambda' = 3\kappa^2\lambda^2 I(m^2)$$

$$\left(I(-p^2) \rightarrow I_{sub}(-p^2)\right)$$

Power Counting:

Feynman diagrams lead to integrals of the form $I \sim \int \frac{dl^D\, Q(l)}{P(l)}$

UV divergences are polynom. in p, m in $P(l)$ may be neglected, α come in as overall factors

$$\frac{dI}{d\log\Lambda} \in \mathrm{Poly}(\alpha_k, p_k, m_k)$$

$$[\mathcal{L}] = D \rightarrow \mathcal{L} = \sum_k{}' \alpha_k \mathcal{L}_k$$

$[\mathcal{L}_k] \leq D$

$[\alpha_k] \geq 0$

$\mathrm{Poly}(\alpha_k, p_k, m_k)$ has non-negative mass dimensions and usually $[I] \leq D$ → there are only few local counterterms determined by the polynomial in p_k, these have mass dimension $\leq D$

Renormalizability: The question is whether all physical quantities remain finite for $\Lambda_{cut} \rightarrow \infty$. Are there enough bare param. to absorb all divergences?

- QFT models where all divergences can be absorbed: renormalizable
- we may introduce further terms and couplings in the Lagrangian to absorb all divergences → finitely many → renormalizable

Running Coupling: We now remove Λ_{cut} by $\Lambda_{cut} \rightarrow \infty$, $\mu(m, \Lambda_{cut}) \rightarrow \infty$, m = const.

$$\frac{d\mu}{d\Lambda_{cut}} = \frac{m\kappa^2}{16\pi^2\Lambda_{cut}} + \dots \leftrightarrow \frac{d\log\mu}{d\log\Lambda_{cut}} = \frac{\kappa^2}{16\pi^2} + \dots \rightarrow \mu \sim m\left(\frac{\Lambda_{cut}}{m}\right)^{\kappa^2/16\pi^2}$$

Similar effects exist for coupling constants:

$$\lambda \sim \left(\frac{p}{\Lambda}\right)^{\propto\kappa^2 + \dots}$$

→ different effective coupling strengths measured depending on the length/energy scale that was probed

→ can be attributed to charges → charge screening